EJTP **11, No. 31** (2014) 71–88 *Electronic Journal of Theoretical Physics*

A Fisher-Bohm Geometry for Quantum Information

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Received 15 June 2014, Accepted 19 August 2014, Published 15 September 2014

Abstract: The underlying geometry of Bohms quantum potential is explored as a new approach to quantum information. If we express the entropy of a quantum system as a superposition of Boltzmann entropies, Bohms quantum potential, under the constraint of a minimum condition of Fisher information, appears related to a Weyl-like gauge potential and the quantum information emerges as a deformation of the quantum system geometry, according to B. Hiley and the etymological meaning of information: the activity of shaping or putting form into a given process. For this purpose, it will be important to study the quantum potential as an expression of quantum entropy in the light of the metric of Fisher-Weyl. Finally, we will examine the geometry of the double-slit interference and the Aharonov-Bohm effect.

Keywords: Quantum Potential; Fisher Information; Quantum Geometry; Weyl-like gauge; Double Slit Interferometer ; Aharonov-Bohm effect
PACS (2010): 03.65.-w; 04.60.Pp; 03.67.-a; 03.67.Ac; 89.70.-a; 03.67.Bg; 03.67.Mn; 89.70.Cf; 02.30.Em

1. Introduction

In the context of classical mechanics, once a classical variable (for examples position or mass) has been measured, it is correct to assume that further measurements will not provide any new information. In the case of the localization, for example, the object will continue to be at its position owing to the fact that the object can, in principle, be isolated from the environment.

In quantum mechanics, the situation is more complex, because the quantum state could be pure or mixed, and these two cases are very different from the point of view of measurement. Moreover, in general, one does not know whether the state is pure or

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mixed. The game for the receiver is to determine this state as closely as possible, after examining as many copies of the state as it is required. The entropy can be thus sees as the average information communicated about the unknown state at any point in the measurement process.

A mixed state is a statistical mixture of component pure states, and its entropy is determined by the von Neumann measure in a manner that is similar to the entropy for classical states. A pure state is completely described by its state function and its von Neumann entropy is zero.

Several theories have been advanced that assign a finite entropy to matter (see, for example, the fundamental work of J. Bekenstein [1]). The finite value of entropy for a given volume has been taken to mean that matter cannot be subdivided infinitely, and that the fundamental entity relating to matter is a bit (1 or 0) of information. However, this approach of discretization has not been very successful. Part of the fault may lie in the limitations of the current concept of quantum entropy. In particular, von Neumann's definition of entropy does not provide the right measure in the asymmetric situation where the choice of the state itself carries information. Von Neumann's definition of quantum entropy seems to meet problems in the interpretation of quantum information as pure states. In this regard, an alternative approach has been proposed by S. Kak that includes thermodynamic information of the pure states[2].

In this article, we suggest a non-Euclidean geometry as the source of a quantum information that derives, on quantum level, from a quantum entropy which is defined as a vector of superposition of different Boltzmann entropies. In this approach, the difference between classical and quantum information is similar to the difference between Euclidean and non-Euclidean geometry in the parameter space determined by the quantum entropy. Superposition states characteristic of quantum mechanics are determined by the deformation of the geometry of the background and are associated with a vector of superposed entropies. In this way we try to reconcile the relational definition of von Neuman with the thermodynamics of Kak in a synthesis which characterizes the geometric deformation of quantum potential as a measure of quantum information, in the spirit of Einstein [3].

The structure of this article is the following: in chapter 2 we make some considerations about von Neumann and Kak approaches to quantum entropy. In chapter 3 we introduce the most important features of Bohm's quantum potential, and in chapter 4 analyze the quantum potential in terms of Fisher information and Weyl geometry based on a definition of quantum entropy as a superposition of different Boltzmann entropies. Finally, in chapters 5 and 6 we will apply the theoretical framework of quantum geometry developed to two physical problems, namely the double-slit interference and the Aharonov-Bohm effect.

2. About von Neumann's and Kak's Quantum Entropy

In order to speak about quantum information, one can assume that the source and the receiver use the same basis vectors for the representation and the measurement of the states (this assumption is necessary to establish the baseline of the game between the source and the receiver). For a quantum system characterized by the density operator ρ, the average information the experimenter obtains in the repeated observations of the very many copies of an identically prepared mixed state is given by the von Neumann entropy

$$S_n(\rho) = \sum_x \lambda_x \ln \lambda_x \tag{1}$$

where λ_x are the eigenvalues of the density matrix associated with the system. In particular, in the case of the mixed state described by the matrix

$$\rho = \begin{bmatrix} p & 0 \\ 0 & 1-p \end{bmatrix}, \tag{2}$$

the entropy (1) becomes

$$S_n(\rho) = -p \ln p - (1-p)\ln(1-p). \tag{3}$$

Equation (3) implies that the von Neumann entropy of a pure state is zero, indicating that once it has been identified then there is no further information to be obtained from its copies, which is not the case with a mixed state. Since an unknown pure state will communicate real information to the receiver, the von Neumann entropy cannot be considered as a reasonable measure of quantum information as regards pure states.

Considering the information transfer problem from the point of view of the preparer of the state and the experimenter, it is clear that both mixed and pure states provide information to the experimenter. In this regard, a detailed analysis has been provided recently by Kak: "*For a two-component elementary mixed state, the most information in each measurement is one bit, and each further measurement of identically prepared states will also be one bit. For an unknown pure state, the information in it represents the choice the source has made out of the infinity of choices related to the values of the probability amplitudes with respect to the basis components of the receiver's measurement apparatus. The maximum information in a pure state is thus infinite. On the other hand, each measurement of a two-component pure state can provide one bit of information. But if it is assumed that the source has made available an unlimited number of identically prepared states, the receiver can obtain additional information from each measurement until the probability amplitudes have been correctly estimated. Once that has occurred, unlike the case of a mixed state, no further information will be obtained from testing additional copies of this pure state. The receiver can do this by adjusting the basis vectors so that he gets closer to the unknown pure state. As the adjustment proceeds, the amount of information that he would obtain from each measurement will decrease. The information that can be obtained from such a state in repeated experiments is potentially infinite in the most general case. But if the observer is told what the pure state is, the information associated with the states vanishes, suggesting that a fundamental divide exists between objective and* *subjective information. [...] One can speak of information associated with a system only in relation to an experimental arrangement together with the protocol for measurement. The experimental arrangement is thus integral to the amount of information that can be obtained.*" [2].

In order to conciliate the fact that, according to the von Neumann the entropy for an unknown pure state is zero with the fact that repeated measurements on copies of such a pure state do communicate information, Kak has proposed a measure for the informational entropy of a quantum state that includes information in the pure states and the thermodynamic entropy. In Kak's proposal the origin of information is explained in terms of an interplay between unitary and non-unitary evolution. More precisely, the starting idea of Kak's model is that the informational entropy of a quantum system with density matrix ρ is given by relation

$$S_i\left(\rho\right) = -\sum_i \rho_{ii} \ln \rho_{ii}. \tag{4}$$

The entropy (4) indicates the average uncertainty that the receiver has in relation to the quantum state *for each measurement.* Should the manner of the preparation of the pure state be known to the observer, he can choose a basis state function that would completely describe it, and there would indeed be no information associated with it.

By appropriately adjusting the basis vectors, the receiver can change the value of this entropy. The value of $S_i\left(\rho\right)$ is the amount of entropy of the quantum system that is accessible to the receiver.

Some properties of Si are the following:

1. $Si\ (\rho) \quad \geq Sn(\rho)$, and the two are equal only when the density matrix has only diagonal terms.

2. $Sn(\rho)$ is obtained by minimizing $Si\ (\rho)$ with respect to all possible unitary transformations.

3. The maximum value of Si is infinity, true for the case where the number of components is infinite.

It is interesting to remark that the informational entropy introduced by Kak can resolve the puzzle of entropy increase in the universe. It assumes that the universe had immensely large informational entropy namely was in a low-entropy quantum state in the beginning, and that then, during the physical evolution of the universe, this informational entropy was transformed into thermodynamic entropy and thus produced high-entropy quantum states, in part because of the second law of thermodynamics, in part because of the expansion of the universe, and above all because non-unitary evolutions of pure states. Given the fact that we have both unitary, U, and non-unitary, Mi , or measurement, operators, the density operator for each elementary state will change either to:

$$\left|\phi\right\rangle_{new} = \begin{cases} U\left|\phi\right\rangle \\ \frac{M_i\left|\phi\right\rangle}{\sqrt{\left\langle\varphi\right|M_i^{+}M_i\left|\phi\right\rangle}} \end{cases} \tag{5}$$

where the first regards unitary evolution and the second non-unitary evolution When only non-unitary operators are used for the evolution, the elementary state will change from the pure state $|\rangle = \alpha\,|\rangle + \beta\,|\rangle$ to the mixed state given by the density matrix:

$$G = \begin{bmatrix} |\alpha|^2 & 0 \\ 0 & |\beta|^2 \end{bmatrix}. \tag{6}$$

Its informational entropy would then have transformed completely from that of the pure state to that of the mixed state, and its von Neumann entropy would now be finite.

The existence of non-unitary operators requires the presence of low-entropy structures. This view of the problem of how information increases is to postulate non-unitary evolution as a part of the earliest universe and requires the framework of dissipative quantum field theory [4].

3. Bohm's Quantum Potential as Active Information

In his classic papers of 1952 David Bohm showed that if one interprets each individual physical system as composed by a corpuscle and a wave guiding it, by writing its wave function in polar form and decomposing the Schrödinger equation, the movement of the corpuscle under the guide of the wave happens in agreement with a law of motion which assumes the following form

$$\frac{\partial S}{\partial t} + \frac{|\nabla S|^2}{2m} - \frac{\hbar^2}{2m}\frac{\nabla^2 R}{R} + V = 0, \tag{7}$$

where $R$is the amplitude and $S$is the phase of the wave function, $\hbar$ is Planck's reduced constant, $m$is the mass of the particle and $V$is the classical potential. This equation is the classical equation of Hamilton-Jacobi, except for the appearance of the additional term

$$Q = -\frac{\hbar^2}{2m}\frac{\nabla^2 R}{R}, \tag{8}$$

having the dimension of an energy, containing Planck constant and therefore appropriately defined quantum potential [5]. In analogous way, in the case of a many-body system, if we consider a wave function $\psi = R\left(\vec{x}_1, ..., \vec{x}_N, t\right) e^{iS(\vec{x}_1,...,\vec{x}_N,t)/\hbar}$, defined on the configuration space R^{3N} of a system of N particles, the movement of this system under the action of the wave ψ happens in agreement to the law of motion

$$\frac{\partial S}{\partial t} + \sum_{i=1}^{N} \frac{|\nabla_i S|^2}{2m_i} + Q + V = 0 \tag{9}$$

where

$$Q = \sum_{i=1}^{N} -\frac{\hbar^2}{2m_i}\frac{\nabla_i^2 R}{R} \tag{10}$$

is the many-body quantum potential.

In virtue of the features of the quantum potential, the basic equations (5) and (9) of non-relativistic Bohm theory do not imply a classical treatment of quantum processes. The non-locality emerges as a fundamental trait of quantum theory.In the expression of the quantum potential, the appearance of the amplitude of the wave function in the denominator also explains why the quantum potential can produce strong long-range effects that do not necessarily fall off with distance and so the typical properties of entangled wave functions. Thus even though the wave function spreads out, the effects of the quantum potential need not necessarily decrease This is just the type of behaviour required to explain the EPR paradox. In virtue of the quantum potential, Bohm's interpretation of quantum phenomena has the merit to include non-locality *ab initio* rather than to come upon it as an *a posteriori* statistical "mysterious weirdness".

Moreover, if we examine the expression of the quantum potential in the double-slit experiment, we find that it depends on the width of the slits, their distance apart and the momentum of the particle. This means that the quantum potential has a contextual nature, namely brings a global information on the process and its environment by individuating an infinite set of phase paths; and it has an active information in the sense that it modifies the behaviour of the particle [6]. In a double-slit experiment, if one of the two slits is closed the quantum potential changes, and this information arrives instantaneously to the particle, which behaves as a consequence. The active information of the quantum potential ((6) or (10)) is deeply different from classical one: it is, in fact, intrinsically not-Shannon-Turing computable[7, 8].

The quantum potential indicates a source of active information internal to the system and differently accessible according to the operations of preparation and state selection, environment and measurement. The quantum system's active information is defined by an infinite uncountable set of phase paths and its very nature is non-local. In such configurations as "quantum gates", based upon a generalization of Turing scheme, the constrains of reversibility and unitarity limit the possibility to detect quantum information just to the outputs of superposition states; and yet nothing prevents our thinking of a different approach to the system's geometry which, within peculiar experimental arrangements, can get qualitatively different answers and endowed with oracular skills, so turning into resource all non-locality features, even those which are traditionally regarded as a limit within the classical scheme, such as de-coherence, dissipation and probabilistic responses [9].

An essential key for the relations between quantum potential, system's geometry and information is provided by the Fisher information [10,11]. The Fisher information can be interpreted as the information an observable random variable X carries about a not-observable parameter θ which the probability distribution X depends on. Such statistical measurement has aroused interest in relation to the study of the distributions of the observables of a quantum system.

As for the physical meaning of the Fisher information, instead, there are very controversial viewpoints. Roy Frieden's programme [12] to derive physics' fundamental equations from an extreme physical Fisher information principle as optimization (or saturation) of the observer/observed relationship has been widely criticized because of its vagueness. Actually, although Frieden's position is epistemologically correct for a good experimental physicist as he is, the principle itself is too less constraining, so that the significant physical features of the systems under observation have to be introduced *ad hoc* in order to make it really effective [13]. Thus, Frieden's programme looks more like a request for coherence between formal structures and distributions of observables than an out-and-out "fundamental principle". The studies where an attempt is made to connect Fisher information with the specific structural aspects of quantum mechanics and to consider it as a statistical indicator of the relationships between classical and quantum information seem much more interesting [14, 15, 16, 17, 18]. In spite of the "interpretative dilemmas", quantum mechanics shows the highest operational nature of any other physical theory, and it is thus greatly interesting that the "thin" statistical distribution of a quantum system can be derived from the quantum potential. In the next chapters, we will show that the Fisher information plays the role of a natural tile to build a metric able to connect the system's statistical outcomes and its global geometry.

4. The Quantum Potential as Fisher Metrics on Entropy Manifold

On the basis of an extension of the tensor calculus to operators represented by non-quadratic matrices [11, 21], it is possible to provide a new interesting geometrical reading in which the quantum potential emerges as active information determined by the vector of the superposition of Boltzmann entropies

$$\begin{cases} S_1 = k \log W_1(\theta_1, \theta_2, ..., \theta_p) \\ S_2 = k \log W_2(\theta_1, \theta_2, ..., \theta_p) \\ ... \\ S_n = k \log W_n(\theta_1, \theta_2, ..., \theta_p) \end{cases} \tag{11}$$

where W are the number of the microstates for the same parameters θ as temperatures, pressures, etc.. . .. In this picture, quantum effects are equivalent to a geometry which is described by the following equation

$$\frac{\partial}{\partial x^k} + \frac{\partial^2 S_j}{\partial x^k \partial x^p} \frac{\partial x^i}{\partial S_j} = \frac{\partial}{\partial x^k} + \frac{\partial \log W_j}{\partial x_h} = \frac{\partial}{\partial x^k} + B_h \tag{12}$$

where B_h is a Weyl-like gauge potential [19, 20]. In this picture, we have a deformation of the moments for the change of the geometry stated by the following expression of the action:

$$\begin{aligned} A &= \int \rho[\tfrac{\partial A}{\partial t} + \tfrac{1}{2m}(p_i + B_i)(p_j + B_j) + V]dtd^n x \\ A &= \int \rho[\tfrac{\partial A}{\partial t} + \tfrac{1}{2m}(p_i p_j + B_i B_j) + V]dtd^n x \\ A &= \int \rho[\tfrac{\partial A}{\partial t} + \tfrac{1}{2m}(p_i p_j + \tfrac{\partial \log W}{\partial x_i}\tfrac{\partial \log W}{\partial x_j}) + V]dtd^n x \\ &= \int \rho[\tfrac{\partial A}{\partial t} + \tfrac{1}{2m}p_i p_j + V)dtd^n x + \tfrac{1}{2m}\tfrac{\partial \log W}{\partial x_i}\tfrac{\partial \log W}{\partial x_j})]dtd^n x \end{aligned} \quad . \tag{13}$$

The quantum action assumes the minimum value when

$$\delta A = 0$$

$$For$$

$$\delta \int \rho[\tfrac{\partial A}{\partial t} + \tfrac{1}{2m}p_i p_j + V)dtd^n x + \delta \tfrac{1}{2m}\tfrac{\partial \log W}{\partial x_i}\tfrac{\partial \log W}{\partial x_j})]dtd^n x = 0 \tag{14}$$

$$so$$

$$\tfrac{\partial A}{\partial t} + \tfrac{1}{2m}p_i p_j + V + \tfrac{1}{2m}(\tfrac{1}{W^2}\tfrac{\partial W}{\partial x_i}\tfrac{\partial W}{\partial x_j} - \tfrac{2}{W}\tfrac{\partial^2 W}{\partial x_i \partial x_j}) = \tfrac{\partial S}{\partial t} + \tfrac{1}{2m}p_i p_j + V + Q$$

where Q is the Bohm quantum potential that is a consequence for the minimum condition of Fisher information [21]. On the basis of equation (14), one can interpret Bohm's quantum potential as an information channel determined by the functions W defining the number of microstates of the physical system under consideration, which depend on the parameters θ of the distribution probability (and thus, for example, on the space-temporal distribution of an ensemble of particles, namely the density of particles in the element of volume d^3x around a point $\vec{x}$ at time t) and which correspond to the vector of the superpose Boltzmann entropies (11). In other words, the distribution probability of the wave function is linked to the functions W defining the number of microstates of the physical system. The quantum entropy emerges from these functions W given by equations (11), and can be considered as the fundamental physical entities which determine the action of the quantum potential (in the extreme condition of the Fisher information) on the basis of equation

$$Q = \frac{1}{2m}\left(\frac{1}{W^2}\frac{\partial W}{\partial x_i}\frac{\partial W}{\partial x_j} - \frac{2}{W}\frac{\partial^2 W}{\partial x_i \partial x_j}\right). \tag{15}$$

Under the constraint of the minimum condition of Fisher information, each "path" of the quantum potential is connected to the entropy by the functions W. In other words, each of the entropies appearing in the superposition vector (11) can be considered as a specific information channel of the quantum potential.

Moreover, on the basis of equations (12) and (14), one can say that the change of the geometry in the presence of quantum effects, which is expressed by a Weyl-like gauge potential is determined by the functions W and thus by the quantum entropy. Therefore, in non relativistic bohmian quantum mechanics the distribution probability of the wave function determines the functions W, the number of microstates of the system into consideration. The quantum entropy emerges from these functions W given by equations

(11), and determine a change of the geometry expressed by a Weyl-like gauge potential and characterized by a deformation of the moments given by equation (13).

Now, by introducing the definition (15) of the quantum potential as an geometric informational entity inside the quantum Hamilton-Jacobi equation, we obtain

$$\frac{|\nabla S|^2}{2m} + V + \frac{1}{2m}\left(\frac{1}{W^2}\frac{\partial W}{\partial x_i}\frac{\partial W}{\partial x_j} - \frac{2}{W}\frac{\partial^2 W}{\partial x_i \partial x_j}\right) = -\frac{\partial S}{\partial t} \tag{16}$$

which provides a new way to read the energy conservation law in quantum mechanics. In equation (16) two quantum corrector terms appear in the energy of the system, which are owed to the functions W linked with the quantum entropy, and which thus describe the change of the geometry in the presence of quantum effects. These two quantum corrector terms can thus be interpreted as a sort of degree of chaos of the background space determined by the ensemble of particles associated with the wave function under consideration. This opens the QM to the QFT and seems to suggest that the latter is the only authentic "realistic interpretation of QM [22]. It is also interesting to observe that the inverse square root of the quantity

$$L_{quantum} = \frac{1}{\sqrt{\frac{1}{\hbar^2}\left(\frac{2}{W}\frac{\partial^2 W}{\partial x_i \partial x_j} - \frac{1}{W^2}\frac{\partial W}{\partial x_i}\frac{\partial W}{\partial x_j}\right)}} \tag{17}$$

defines a typical quantum-entropic length that can be used to evaluate the strength of quantum effects and, therefore, the modification of the geometry with respect to the Euclidean geometry characteristic of classical physics. Once the quantum-entropic length becomes non-negligible the system goes into a quantum regime. In this picture, Heisenberg's uncertainty principle derives from the fact that we are unable to perform a classical measurement to distances smaller than the quantum-entropic length. So, the size of a measurement has to be bigger than the quantum-entropic length:

$$\Delta L \geq L_{quantum} = \frac{1}{\sqrt{\frac{1}{\hbar^2}\left(\frac{2}{W}\frac{\partial^2 W}{\partial x_i \partial x_j} - \frac{1}{W^2}\frac{\partial W}{\partial x_i}\frac{\partial W}{\partial x_j}\right)}}. \tag{18}$$

The quantum regime is entered when the quantum-entropic length must be taken under consideration.

Novello, Salim and Falciano [20] have recently proposed a geometrical approach in which the presence of quantum effects is linked with the Weyl length $L_W = \frac{1}{\sqrt{R}}$, and thus with the curvature scalar; in analogous way, in the approach here proposed, the quantum effects are owed to the microstates characterizing the system under consideration and thus the vector of the superposition of different Boltzmann entropies (11) emerges thus as the most fundamental source of quantum information.

5. The Geometry of Quantum Mechanics Interference

As Feynman stated, the double-slit interference "... has in it the heart of quantum mechanics. In reality, it contains the only mystery" [23]. It is thus the appropriate starting-point to apply the entropic approach to quantum potential developed in chapter 4 and to illustrate the concept of quantum information as a measure of the deformation of the quantum entropy space.

Let us consider a wave function characterized by N probability densities $h_1, h_2, ..., h_n$:

$$|\psi\rangle = |h_1\rangle + |h_2\rangle + ... + |h_n\rangle \tag{19}$$

where $h_1 = \alpha_1 + i\beta_1$, $h_2 = \alpha_2 + i\beta_2$, $\ldots$, $h_n = \alpha_n + i\beta_n$.

The probability for the interference is

$$P(x) = \langle \psi \mid \psi \rangle = \sum_{i,j} g_{ij} \xi^i(x) \xi^j(x) \tag{20}$$

where

$$\xi^i = \sqrt{I^i}, \quad \xi^j = \sqrt{I^j} \tag{21}$$

and

$$g = \begin{bmatrix} 1 & \cos(\alpha_1 - \alpha_2) & ... & \cos(\alpha_1 - \alpha_2) \\ \cos(\alpha_2 - \alpha_1) & 1 & \cos(\alpha_2 - \alpha_1) & \cos(\alpha_2 - \alpha_1) \\ ... & ... & ... & ... \\ \cos(\alpha_n - \alpha_1) & \cos(\alpha_n - \alpha_2) & ... & 1 \end{bmatrix}. \tag{22}$$

On the basis of the metric (22), the square-distance between the end-points of two vectors ξ^i and η^i is

$$s^2(\theta_k) = \sum_{i,j} g_{i,j} \left(\xi^i(\theta_k) - \eta^i(\theta_k)\right)\left(\xi^j(\theta_k) - \eta^j(\theta_k)\right). \tag{23}$$

Moreover, when $\eta^i = \xi^i + \frac{\partial \xi^i}{\partial \theta_k}$ we obtain

$$ds^2 = \sum_{i,j} g_{i,j} \left(\frac{\partial \xi^i}{\partial \theta_k} \partial \theta_k\right)\left(\frac{\partial \xi^i}{\partial \theta_h} \partial \theta_h\right) = G_{h,k} \partial \theta^h \partial \theta^k \tag{24}$$

where

$$G = A^T g A \tag{25}$$

and

$$A = \begin{bmatrix} \frac{\partial \xi^1}{\partial \theta_1} & \frac{\partial \xi^1}{\partial \theta_2} & \cdots & \frac{\partial \xi^1}{\partial \theta_q} \\ \frac{\partial \xi^2}{\partial \theta_1} & \frac{\partial \xi^2}{\partial \theta_2} & \cdots & \frac{\partial \xi^2}{\partial \theta_q} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial \xi^n}{\partial \theta_1} & \frac{\partial \xi^n}{\partial \theta_2} & \cdots & \frac{\partial \xi^n}{\partial \theta_q} \end{bmatrix}. \tag{26}$$

On the basis of equations (24), (25) and (26), we have n quantum states and n parameters of the states in order to describe the process of interference [11].

Now, in the extreme condition of Fisher information given by equation (14), we can introduce the quantum-entropic length given by equation (17) inside equation (24) and thus we obtain:

$$ds^2 = G_{h,k}\partial\theta^h\partial\theta^k = \frac{1}{\frac{1}{\hbar^2}\left(\frac{2}{W}\frac{\partial^2 W}{\partial x_i \partial x_j} - \frac{1}{W^2}\frac{\partial W}{\partial x_i}\frac{\partial W}{\partial x_j}\right)}. \tag{27}$$

Equation (27) indicates clearly that the parameter states characterizing the quantum states in the process of the interference are determined by the functions W and thus by the quantum entropies. Incidentally, the (27) also suggests that in Physics the concept of distance always means a distinguishability between physical states. The probability distribution associated with the wave function it is fixed by the functions W defining the number of microstates of the beams of electrons. The interference process is associated with the quantum entropic length (27) and thus is determined by the quantum entropy. The vector of the superposed quantum entropies can be considered thus the ultimate physical reason in the experiment of the double-slit interference.

6. The Aharonov-Bohm Quantum Geometry

The second process we consider is the Aharonov-Bohm setup where a charged particle confined to a box acquires a geometric phase while slowly taking the box around a magnetic flux. The Aharonov-Bohm effect involves the quantum mechanical scattering of electrons in the presence of a classical magnetic vector potential produced by a current-carrying solenoid. Electrons are prevented from entering the region where the magnetic field itself is nonzero so there is no classical force on them. Nevertheless, information about the flux can be obtained.

As Aharonov and Bohm [24] showed, information about the magnetic flux (modulo a constant) through a solenoid can be measured via its effect on the interference pattern of electrons in a two-slit experiment. A particle of charge q traversing a path $C_\pm$ in the presence of a magnetic vector potential $\vec{A}$ can be described by a wave function $\psi_\pm(x)\, e^{\frac{iq}{\hbar c}\int_{C\pm}\vec{A}\cdot d\vec{l}}$ where $\psi_\pm(x)$ is the wave function in the absence of a vector potential and the integral is taken along the corresponding path $C_\pm$. We define the flux parameter φ as

$$\varphi = \frac{q}{\hbar c}\left(\int_{C_+}\vec{A}\cdot d\vec{l} - \int_{C_-}\vec{A}\cdot d\vec{l}\right) = \frac{q}{\hbar c}\oint_C \vec{A}\cdot d\vec{l} = \frac{q}{\hbar c}\Phi \tag{28}$$

where C is a closed curve created by following C+ and returning along C- and Φ is the flux through the solenoid, and thus through any closed surface bounded by C. In terms of these variables, the total wavefunction at the screen is

$$\psi(x,\varphi) = e^{\frac{iq}{\hbar c}\int_{C_+}\vec{A}\cdot d\vec{l}}\left(\psi_+(x) + \psi_-(x)\, e^{-i\varphi}\right) \tag{29}$$

which is gauge invariant up to an overall phase.

To be more explicit, let us consider, for example, two wave packets A and B of an electron that passes on the two sides of a solenoid containing a magnetic flux Φ. On the basis of equation (5), when the line AB passes the solenoid, the wave packet A acquires a phase difference $\alpha = \frac{e}{\hbar c}\Phi$ with respect to the wave packet B. As a consequence, the expectation value of the modular momentum becomes $\exp\left(i\frac{e}{\hbar c}\Phi\right)/2$ and this change in the expectation value of the modular momentum by the factor $\exp\left(i\frac{e}{\hbar c}\Phi\right)/2$ as the line AB joining the two wave packets crosses the solenoid is the same in every gauge.

In the recent article *Quantum Measurement and the Aharonov-Bohm Effect* with Superposed Magnetic Fluxes, Bradonijc and Swain have considered what happens to the Aharonov-Bohm effect if the flux is in superposition of two states with different classical values [25]. These two authors have found that the interference pattern in the Aharonov-Bohm effect contains information about the nature of the superposition, allowing information about the state of the flux to be extracted without disturbing it. Bradonijc's and Swain's put in evidence that the information is obtained by a non-local operation Involving the vector potential without transfer of energy or momentum.By assuming that the current in the solenoid is in a superposition of two macroscopic states corresponding to equal and opposite currents, resulting in a superposition of positive and negative magnetic flux inside the solenoid, since the probability amplitude for a particle to be found at position x on the screen flux up due to vector potential $\vec{A}_\uparrow$ is

$$\psi_\uparrow(x,\varphi) = e^{\frac{iq}{\hbar c}\int\limits_{C_+}\vec{A}_\uparrow\cdot d\vec{l}}\left(\psi_+(x) + \psi_-(x)\,e^{i|\varphi|}\right) \tag{30}$$

and for the same flux down due to a vector potential $\vec{A}_\downarrow$ is

$$\psi_\downarrow(x,\varphi) = e^{\frac{iq}{\hbar c}\int\limits_{C_+}\vec{A}_\downarrow\cdot d\vec{l}}\left(\psi_+(x) + \psi_-(x)\,e^{-i|\varphi|}\right) \tag{31}$$

the general wave function of the particle is

$$|\psi(x,\varphi)\rangle = Cos\frac{\varphi}{2}\,|\psi_\uparrow\rangle + Sin\frac{\varphi}{2}e^{i\omega}\,|\psi_\downarrow\rangle \tag{32}$$

where $0 \le \vartheta \le \pi$ and $0 \le \omega \le 2\pi$. On the basis of the wave function (32), the total probability density at the screen is

$$|\psi(x,\varphi)|^2 = |\psi_+|^2 + |\psi_-|^2 + 2R\left(\psi_+^*\psi_-\right)\cos|\varphi| - 2J\left(\psi_+^*\psi_-\right)\sin|\varphi|\left(\cos^2\left(\frac{\vartheta}{2}\right) - \sin^2\left(\frac{\vartheta}{2}\right)\right). \tag{33}$$

It is interesting to remark that Bradonijc's and Swain's approach can be considered "as a toy model for the quantum mechanical propagation of a particle in a background spacetime which is a superposition of different classical geometries" [25].

Now, the quantum mechanical propagation of a particle in a background which is a superposition of different classical geometries, can be indeed seen, inside the geometrodynamics approach of the quantum potential in entropy space here suggested, as a consequence of the most general state of the background space given by the superposition of different Boltzmann entropies. The vector of the superposed entropies (11) can be indeed seen as the fundamental entity which determines the quantum mechanical propagation of a particle in a background, given by a superposition of different classical geometries, characteristic of the Aharonov-Bohm effect. This result can be shown by appropriately including the quantum potential inside Bradonijc's and Swain's approach of Aharonov-Bohm effect.

In the pilot wave theory, the Aharonov-Bohm effect is explained through the local but indirect action of the vector potential on the particle via the quantum force [26, 27]. The Hamilton-Jacobi equation and the force law for this problem are

$$\frac{\partial S}{\partial t} + \frac{|\nabla S|^2}{2m} + eA_0 + Q = 0 \tag{34}$$

$$m\frac{d\vec{v}}{dt} = -\nabla Q + \vec{F} \tag{35}$$

where $\vec{F}$ is the Lorentz force, and the physical momentum $m\vec{v} = \nabla S - \frac{e}{c}\vec{A}$ and Q are gauge invariant quantities. Even when $\vec{F} = 0$ the quantum potential is modified by $\vec{A}$. The latter may therefore be expected to cause a redistribution of the trajectories in the two-slit experiment. In the Aharonov-Bohm effect, the quantum potential may be expressed as

$$Q(\vec{x}, \Phi) = Q_0(\vec{x}) + f(\vec{x}, \Phi) \tag{36}$$

where $Q_0(\vec{x})$ is independent of the flux. The term $Q_0(\vec{x})$ is just determined by the vector of the superposed entropies (11) and thus can be assimilated to equation (15). By introducing the quantum potential (35) in the picture analysed by Bradonijc and Swain, we rewrite the wave function of a particle to be found at position x on the screen flux up due to vector potential $\vec{A}_\uparrow$ (namely the equation (30) as

$$\psi_\uparrow(x, \varphi) = e^{\frac{iq}{\hbar c}\int\limits_{C_+}\left(\vec{A}_\uparrow + Q_0 + f(\vec{x},\Phi)\right)\cdot d\vec{l}}\left(\psi_+(x) + \psi_-(x)\,e^{i|\varphi|}\right) \tag{37}$$

and the wave function for the same flux down due to a vector potential $\vec{A}_\downarrow$ (equation (31)) as

$$\psi_\downarrow(x, \varphi) = e^{\frac{iq}{\hbar c}\int\limits_{C_+}\left(\vec{A}_\downarrow + Q_0 + f(\vec{x},\Phi)\right)\cdot d\vec{l}}\left(\psi_+(x) + \psi_-(x)\,e^{-i|\varphi|}\right) \tag{38}$$

and thus the general wave function of the particle as

$$\psi_\uparrow(x, \varphi) = e^{\frac{iq}{\hbar c}\int\limits_{C_+}\left(\vec{A}_\uparrow + Q_0 + f(\vec{x},\Phi)\right)\cdot d\vec{l}}\left(\psi_+(x) + \psi_-(x)\,e^{i|\varphi|}\right) \tag{39}$$

In equations (37), (38) and (39), the origin of the quantum potential $Q_0(\vec{x})$ is just the vector of the superposed entropies (11), on the basis of the general definition of the quantum potential given by equation (15). Equations (37), (38) and (39) suggest that if the flux is in superposition of two states with different classical values, and thus the background is a superposition of different classical geometries, the quantum potential is modified by the vector potential in the sense that it contains, besides to the term (15) associated with the vector of the superposed entropies, a term depending itself on the flux. The wave functions (37), (38) and (39) indicate that the change of the geometry of physical space determined by the Aharonov-Bohm effect is expressed by the quantities

$$A_{\uparrow} + \frac{1}{2m}\left(\frac{1}{W^2}\frac{\partial W}{\partial x_i}\frac{\partial W}{\partial x_j} - \frac{2}{W}\frac{\partial^2 W}{\partial x_i \partial x_j}\right) + f(\vec{x}, \Phi) \tag{40}$$

and

$$A_{\downarrow} + \frac{1}{2m}\left(\frac{1}{W^2}\frac{\partial W}{\partial x_i}\frac{\partial W}{\partial x_j} - \frac{2}{W}\frac{\partial^2 W}{\partial x_i \partial x_j}\right) + f(\vec{x}, \Phi)\,. \tag{41}$$

On the basis of the quantities (40) and (41) describing the deformation of the geometry of physical space in the presence of the Aharonov-Bohm effect, the total probability density at the screen, given by equation

$$|\psi(x,\varphi)|^2 = |\psi_+|^2 + |\psi_-|^2 + 2R\left(\psi_+^*\psi_-\right)\cos|\varphi| - 2J\left(\psi_+^*\psi_-\right)\sin|\varphi|\left(\cos^2\left(\frac{\vartheta}{2}\right) - \sin^2\left(\frac{\vartheta}{2}\right)\right) \tag{42}$$

where ψ_+ and ψ_- are derived from equations (37) and (38), turns out to be determined just by these quantities (40) and (41).

Let us analyse now some interesting results which derive from equation (42) for various choices of ϑ.

1. For $\vartheta = 0$, $|\psi(x,\varphi)|^2$ goes to regular Aharonov-Bohm effect with an "up" flux.
2. For $\vartheta = \pi$, $|\psi(x,\varphi)|^2$ goes to regular Aharonov-Bohm effect with a "down" flux.
3. For $\vartheta = \pi/2$, the last term in equation (42) goes to zero. There is still an interference pattern, but it is different from the interference patterns in cases 1 and 2 above.

The first two limits indicate that that equation (42) reduces to the expected expressions for two classical flux states. More interesting is the third limit, which indicates that it is in principle possible to extract information about the quantum mechanical state of magnetic flux in the Aharonov-Bohm experiment from the electron diffraction pattern without disturbing the state of the flux. This information is extracted via a fundamentally nonlocal operation involving the deformation of physical space determined by the quantum potential and the vector potential (associated with the quantities (40) and (41)) and over an extended region of spacetime, and without any interaction in the region where the (superposition of) classical magnetic fields is present (i.e. the excluded region inside the solenoid). In other words, according to the geometry of quantum entropy space here suggested, the deformation of the physical space expressed by the quantities (40) and (41) can be seen as the source of information in the Aharonov-Bohm effect.

Moreover, in analogy with what we have seen in chapters 4 and 5 as regards the concept of a quantum-entropic length (equations (17) and (27) respectively), also as regards the Aharonov-Bohm effect, in the extreme condition of Fisher information given by equation (14), we can introduce a quantum-entropic length characterizing this quantum phenomenon given by the following equations:

$$L_{Aharonov-Bohm} = \frac{1}{\sqrt{\frac{1}{\hbar^2}\left(\frac{2}{W}\frac{\partial^2 W}{\partial x_i \partial x_j} - \frac{1}{W^2}\frac{\partial W}{\partial x_i}\frac{\partial W}{\partial x_j}\right) - 2m\left(A_{\uparrow} + f(\vec{x}, \Phi)\right)}} \tag{43}$$

(which is the Aharonov-Bohm length associated with the region of physical space generating a flux up on the screen), and

$$L_{Aharonov-Bohm} = \frac{1}{\sqrt{\frac{1}{\hbar^2}\left(\frac{2}{W}\frac{\partial^2 W}{\partial x_i \partial x_j} - \frac{1}{W^2}\frac{\partial W}{\partial x_i}\frac{\partial W}{\partial x_j}\right) - 2m\left(A_{\downarrow} + f\left(\vec{x}, \Phi\right)\right)}} \tag{44}$$

(which is the Aharonov-Bohm length associated with the region of physical space generating a flux down on the screen). Equations (43) and (44) indicate clearly that the features of the background space in the presence of the Aharonov-Bohm effect are determined by the functions W (and thus by the quantum entropies), by the vector potential and the function linked with the phase difference. The Aharonov-Bohm effect is associated with the quantum entropic lengths (43) and (44) and thus is determined by the quantum entropy, the vector potential and the phase difference.

Although, in the Aharonov-Bohm setup, information is obtainable from the interference pattern associated with the deformation of physical space associated with the quantities (40) and (41), it is also important to mention that this information is not complete. In the classical Aharonov-Bohm effect, the flux Φ can only be determined modulo $2\pi\frac{\hbar c}{e}$. For $\Phi = 0$ modulo $2\pi\frac{\hbar c}{e}$, the interference pattern is the same as that for a positive and negative classical flux: there is no effect on the pattern. Superpositions of two magnetic fluxes (determined by the quantities (40) and (41) and thus associated with the geometries expressed by the quantum-entropic lengths (43) and (44)) which would individually be detectable via the Aharonov-Bohm effect can give rise to interference patterns which differ from any found in the classical (or non superposed) case.

It is also interesting to remark that the information about superposition is gathered from the full interference pattern. For any finite experimental resolution and finite number of electrons scattered, one can only construct the likelihood that the observed interference pattern corresponds to the prediction for an arbitrary superposition of fluxes. No single electron scattering event provides unambiguous information even about the flux modulo $2\pi\frac{\hbar c}{e}$.

One can also consider the case of fluxes which are detectable by the usual Aharonov-Bohm effect, but with unequal superpositions of "up" and "down" fluxes ($\left|\cos\frac{\vartheta}{2}\right|^2 \neq \left|\sin\frac{\vartheta}{2}\right|^2$). In this case, again, one extracts information about the nature of the state without any interaction which should cause the "collapse" of the state into one of definite flux.

To conclude our geometrodynamics treatment of the Aharonov-Bohm effect in the entropy space, it is finally important to mention the important role of the Berry phase. In his article A new phase in quantum computation Sjökvist has remarked that a geometric quantum information can be obtained that employs one-dimensional geometric phase factors. Sjökvist has written: "*The Berry phase, which occurs in situations like the Aharonov-Bohm setup [. . .] arises in adiabatic evolution, but now for non degenerate eigenspaces of Hamiltonians. Berry phases may be used for quantum computation by encoding the logical states in non degenerate energy levels, such as in the spin-up and spin-down states of a spin -1/2 particle in a magnetic field. When this field rotates* *slowly around a loop, the spin states will pick up Berry phases of magnitude given by half the enclosed solid angle and of opposite sign, which defines an adiabatic geometric phase-shift gate acting on the two spin states*" [28]. The Berry phase in the Aharonov-Bohm effect can be just seen as a consequence of the vector of the superposed entropies, which are indeed the ultimate sources of quantum information, more precisely of the quantities (40) and (41) describing the deformation of the geometry in this process.

Conclusions

In this article a geometric approach to quantum information based on the quantum potential has been proposed. In this approach quantum information can be interpreted as a measure of the deformation of the physical background space determined by the quantum entropy and is associated with a quantum-entropic length. This approach provides a new suggestive geometric perusal of the double-slit interference and of the Aharonov-Bohm effect. It is increasingly recognized the role of emerging properties and global geometry from the collective behavior of qubit states. Apart from the mathematical interest, we believe that a geometric approach to quantum information can complement existing ones and release all the real possibilities of quantum computing [29, 30].

References

[1] Bekenstein, J. D. : Black holes and entropy.In Physical Review D **7**, 2333–2346 (1973).

[2] Kak, S. : Quantum information and entropy.In International Journal of Theoretical Physics, Vol. 46, No. 4, 860-876 (2007)

[3] Yang, C. N. : Einstein's impact on theoretical physics.In Physics Today, Vol. 33, No.6 ,42 (1980)

[4] Blasone, M, Jizba, P. Vitiello, G. : Quantum Field Theory and its Macroscopic Manifestations, Imperial College Press (2011)

[5] Bohm, D. , Hiley, B. J. : The Undivided Universe: an Ontological Interpretation of Quantum Theory. Routledge, London (1993)

[6] Fiscaletti, D.: The geometrodynamic nature of the quantum potential. In Ukr. J. Phys. 2012. Vol. 57, No. 5 , 560-572 (2012)

[7] Licata, I: Emergence and Computation at the Edge of Classical and Quantum Systems. In: Physics of Emergence and Organization, Editors Licata, I. , Sakaji, A., World Scientific Singapore p. 1-25 (2008)

[8] HileyB.J. : From the Heisenberg picture to Bohm: A new perspective on active information and its relation to Shannon information. In Quantum Theory: Reconsidering of Foundations, ed. Khrennikov,A. Vaxjo University Press, pp. 141–162 (2002)

[9] Licata I. : Beyond Turing: Hypercomputation and Quantum Morphogenesis. In Asia Pacific Math. Newslett., vol.2, 20-24 (2012)

[10] Carroll, R. W. : Fluctuations, Information, Gravity and the Quantum Potential, Springer (2006)

[11] Licata I, Resconi G.: Information as Environment Changings. Classical and Quantum Morphic Computation. In: Minati, G., Pessa, E., Abram, M. (Eds): Methods, Models, Simulations and Approaches Towards A General Theory Of Change , World Scientific, Singapore, 47-81 (2012)

[12] Frieden, R.B. :Science from Fisher Information: A Unification. Cambridge University Press (2004)

[13] Streater, R.F. : Lost Causes in and Beyond Physics. Springer (2007)

[14] Reginatto, M.: Derivation of the equations of nonrelativistic quantum mechanics using the principle of minimum Fisher information. In: Phys. Rev. A, vol. 58, n 3, 1775-1778 (1998)

[15] Hall, J. W. M. : Quantum Properties of Classical Fisher Information. In Phys. Rev. A 62, 012107 (2000)

[16] Shun-Long, Luo: Fisher Information of Wavefunctions: Classical and Quantum, Chinese Phys. Lett.,23, 3127-3130 (2006)

[17] Luati, A. : Maximum Fisher information in mixed state quantum systems, Annals of Statistics, 32 , 4, 1770-1779 (2004)

[18] Garbaczewski, P. : Random Dynamics, Entropy Production and Fisher Information. In Acta Phys. Pol. B 34, 3555-3568 (2003)

[19] Castro, C., Mahecha, J. : On Nonlinear Quantum Mechanics, Brownian Motion, Weyl Geometry and Fisher Information. In Progress in Physics, Vol.1, 38-45 (2006)

[20] Falciano, F. T. ,Novello ,M., Salim, J. M. : Geometrizing Relativistic Quantum Mechanics. In Found. Of Phys. 40, 1885–1901 (2010)

[21] Fiscaletti, D., Licata, I. : Weyl Geometries, Fisher Information and Quantum Entropy in Quantum Mechanics**.** In Int J Theor Phys 51:3587–3595(2012)

[22] Preparata, G. : An Introduction to a Realistic Quantum Physics. World Scientific (2002)

[23] Feynman, R. P., Leighton, R. ,Sands, M: The Feynman Lectures on Physics. Addison Wesley (2005)

[24] Aharonov, Y. ,Bohm: Significance of Electromagnetic Potentials in the Quantum Theory. In Phys. Rev. 115, 485–491 (1959)

[25] Bradonjic, K. ,Swain, J. D.: Quantum measurement and the Aharonov-Bohm effect with superposed magnetic fluxes. In arXiv:1103.1607v2 [quant-ph], 2011

[26] Holland, P. R.: The Quantum Theory of Motion: An Account of the De Broglie-Bohm Causal Interpretation of Quantum Mechanics**.** Cambridge Univ. Press (2008)

[27] Bohm, D., Philippidis, C. , Kaye, R. D.: The Aharonov-Bohm effect and the quantum potential.In Nuovo Cimento, **71B**, 75-87 (1982)

[28] Sjöqvist, E. : A New Phase in Quantum Computation. In Physics 1, 35 (2008)

[29] Licata, I: Effective Physical Processes and Active Information in Quantum Computing. In Quantum Biosystems, 1, 51-65 (2007)

[30] Bruno, A., Capolupo, A., Kak, S., Raimondo, G, Vitiello, G.: Berry-Like Phase and Gauge Field in Quantum Computing. In: Minati, G., Pessa, E., Abram, M. (Eds): Methods, Models, Simulations and Approaches Towards A General Theory Of Change , World Scientific, Singapore, 83-94 (2012)