The Origin of Randomness in Quantum Mechanics

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Abstract: A mysterious problem of randomness in quantum mechanics is revisited. This problem being hidden in the Schrödinger equation becomes transparent in its Madelung version. It has been demonstrated that randomness in quantum mechanics has the same mathematical source as that in turbulence and chaos. Special attention is concentrated on equivalence between the Schrodinger and the Madelung equations in connection with the concept of stability in dynamics.

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1. Introduction

Quantum mechanics has introduced randomness into the basic description of physics via the uncertainty principle. In the Schrödinger equation, randomness is included in the wave function. But the Schrödinger equation does not simulate randomness: it rather describes its evolution from the prescribed initial (random) value, and this evolution is fully deterministic. The main purpose of this work is to trace down the mathematical origin of randomness in quantum mechanics, i.e. to find or build a “bridge” between the deterministic and random states. In order to do that, we will turn to the Madelung equation, [1]. For a particle mass $m$ in a potential $F$, the Madelung equation takes the following form

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \left( \frac{\rho}{m} \nabla S \right) = 0 \quad (1)$$

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\[
\frac{\partial \rho}{\partial t} + \frac{1}{2m}(\nabla \rho)^2 + F - \frac{\hbar^2 \nabla^2 \sqrt{\rho}}{2m\sqrt{\rho}} = 0 \tag{2}
\]

Here \(\rho\) and \(S\) are the components of the wave function \(\psi = \sqrt{\rho} e^{iS/\hbar}\), and \(\hbar\) is the Planck constant divided by \(2\pi\). The last term in Eq. (2) is known as quantum potential. From the viewpoint of Newtonian mechanics, Eq. (1) is the equation that expresses continuity of the flow of probability density, and Eq. (2) is the Hamilton-Jacobi equation for the action \(S\) of the particle. Actually the quantum potential in Eq. (2), as a feedback from Eq. (1) to Eq. (2), represents the difference between the Newtonian and quantum mechanics, and therefore, it is solely responsible for fundamental quantum properties.

The Madelung equations (1) and (2) can be converted to the Schrödinger equations using the ansatz

\[
\sqrt{\rho} = \Psi \exp(-iS/\hbar) \tag{3}
\]

where \(\rho\) and \(S\) being real function.

Reversely, Eqs. (1) and (2) can be derived from the Schrödinger equation

\[
i\frac{\partial \Psi}{\partial t} + \frac{\hbar^2}{2m} \nabla^2 \Psi - F\Psi = 0 \tag{4}
\]

using the ansatz, which is reversed to (3)

\[
\Psi = \sqrt{\rho} \exp(iS/\hbar) \tag{5}
\]

So there is one-to-one correspondence between the solutions of the Madelung and the Schrödinger equations. From the stochastic mechanics perspective, the transformation of nonlinear Madelung equation into the linear Schrödinger equation is just a suitable mathematical technique that provides an easy way of finding their solutions.

Before starting the analysis of the Madelung equations, we have to notice that a physical equivalence of the Schrödinger and the Madelung equations is still under discussion.

In the paper [1], T. Wallstrom claims that the description of the particle’s motion as a certain "conservative" diffusion is not equivalent to quantum mechanics in spite of the fact that the Madelung "hydrodynamic" equations, which provide the description of such diffusion, can be converted to the Schrödinger equation. He pointed out that such a stochastic theory can be regarded as equivalent to conventional quantum mechanics only if they can derive from it not just the Madelung equations but also the condition that the circulation of the "probability fluid" is always quantized, which is equivalent the condition for the single-valuedness of the wave function. He claims that to recover the Schrödinger equation, one must add by hand a quantization condition, as in the old quantum theory.

However as shown in [2], the single-valuedness of the wave function required in quantum mechanics, is not an auxiliary condition but a property of all local solutions of the Schrödinger equation. Based on the one-to-one correspondence between local solutions of the Schroedinger and the Madelung equations this means that the quantization of
the circulation of the "probability fluid" is a property of all solutions of the Madelung equations.

Although we incline to support the view expressed in [2], we will not make any further comments on this discussion since our target is the mathematical rather than physical equivalence between these two forms of the quantum formalism. Instead we will make the following comments:

1. In the Schrödinger equation, values of physical observables such as energy and momentum are no longer considered as values of functions on phase space, but as eigenvalues; more precisely: as spectral values of linear operators in Hilbert space.

Unlike that, the state variables of the Madelung equations preserve their classical meaning.

2. The Schrödinger equation does not simulate randomness, but rather describe its evolution in terms of the probability density, and that description is fully deterministic.

Unlike that, as will be demonstrated in this paper, Eq. (2) of the Madelung system simulates randomness: each trajectory described by the solution of this equation occurs randomly with the probability controlled by Eq. (1).

3. Since the concept of stability is not a physical invariant being dependent upon the frame of reference, upon the metric defining the distance between basic and perturbed motions, upon the class of function in which the solution is sought, etc., the solutions of the Schrödinger and Madelung equations may have different criteria of instability.

Now let us divert our attention from the physical interpretation of these equations and consider a formal mathematical problem of solving differential equations (1) and (2) subject to some initial and boundary conditions. In order not to be bounded by the quantum scale, we will assume that \( \hbar \) is not necessarily the Planck constant and it can be any positive number of a classical scale having the dimensionality of action. A particular question we will ask is the following: what happen if we simulate Eqs. (1), and (2) using, for instance, electrical circuits or optical devices, and how will deterministic initial conditions generate randomness that is supposed to be present in the solutions?

2. **Search for transition from determinism to randomness**

Turning to Eq. (2), we will start with some simplification assuming that \( F = 0 \). Rewriting Eq. (2) for the one-dimensional motion of a particle, and differentiating it with respect to \( x \), one obtains

\[
m \frac{\partial^2 x(X,t)}{\partial t^2} - \frac{k^2}{2m} \frac{\partial}{\partial X} \left[ \frac{1}{\sqrt{\rho(X)}} \frac{\partial^2 \sqrt{\rho(X)}}{\partial X^2} \right] = 0 \quad (6)
\]

where \( \rho(X) \) is the probability distribution of \( x \) over its possible values \( X \).

Without the last term, Eq. (6) would represent the second Newton’s law applied to the inertial motions of infinite number of independent samples of a particle forming a continuum \( x(X) \). The last term in Eq. (6), that is a feedback from the Liouville equation, introduces an additional “force” that depends upon the probability distribution of \( x \) over
$X$, and thereby, it couples motions of all possible samples $x(X)$. (It should be noticed that from the viewpoint of usual interpretation of quantum mechanics, Eq. (6) is meaningless since it describes the particle trajectories that “cannot be detected”).

Let us choose the following initial conditions for the deterministic state of the system:

$$x = 0, \quad \rho = \delta(|x| \to 0), \quad \dot{\rho} = 0 \text{ at } t = 0 \tag{7}$$

We intentionally did not specify the initial velocity $\dot{x}$ expecting that the solution will comply with the uncertainty principle.

Now let us rewrite the one-dimensional version of Eqs. (1) and (2) as

$$\frac{\partial^2 \rho}{\partial t^2} + \frac{\hbar^2}{2m^2} \frac{\partial^4 \rho}{\partial X^4} + \xi = 0 \text{ at } t \to 0 \tag{8}$$

where $\xi$ includes only lower order derivatives of $\rho$. For the first approximation, we ignore $\xi$ (later that will be justified,) and solve the equation

$$\frac{\partial^2 \rho}{\partial t^2} + a^2 \frac{\partial^4 \rho}{\partial X^4} = 0 \text{ at } t \to 0 \quad a^2 = \frac{\hbar^2 T^2}{2m^2 L^4} \tag{9}$$

subject to the initial conditions (7). The closed form solution to this problem is known from the theory of nonlinear waves, [3]

$$\rho = \frac{1}{\sqrt{4\pi t \frac{\hbar}{2m}}} \cos\left(\frac{x^2}{4t \frac{\hbar}{2m}} - \frac{\pi}{4}\right) \text{ at } t \to 0 \tag{10}$$

Based upon this solution, one can verify that $\xi \to 0$ at $t \to 0$, and that justifies the approximation (9) (for the proofs see the sub-section 2*). It is important to remember that the solution (10) is valid only for small times, and only during this period it is supposed to be positive and normalized.

Rewriting Eq. (6) in dimensionless form

$$\ddot{x} - a^2 \frac{\partial}{\partial X} \left[ \frac{1}{\sqrt{\rho(X)}} \frac{\partial^2 \sqrt{\rho(X)}}{\partial X^2} \right] = 0 \tag{11}$$

and substituting Eq. (10) into Eq. (11) at $X = x$, after Taylor series expansion, simple differentiations and appropriate approximations, one arrives at the following differential equation instead of (11).

$$\ddot{x} = \frac{c x}{t^2}, \quad c = -\frac{3}{8\pi^2 a^2} \tag{12}$$

This is the Euler equation, and it has the following solution, [4]

$$x = C_1 t^{1+s} + C_2 t^{\frac{1}{s} - s} \text{ at } 4c + 1 > 0 \tag{13}$$

$$x = C_1 \sqrt{t} + C_2 \sqrt{t} \ln t \text{ at } 4c + 1 = 0 \tag{14}$$

$$x = C_1 \sqrt{t} \cos(s \ln t) + C_2 \sqrt{t} \sin(s \ln t) \text{ at } 4c + 1 < 0 \tag{15}$$
where
\[ 2s = \sqrt{|4c + 1|} \]  
(16)

Thus, the qualitative structure of the solution is uniquely defined by the dimensionless constant \( a^2 \) via the constants \( c \) and \( s \), (see Eqs. (12) and (16). But the cases (14) and (15) should be disqualified at once since they are in a conflict with the approximations used for derivation of Eq. (12), (see sub-section 2*).

Hence, we have to stay with the case (13). This gives us the limits
\[ 0 < |c| < 0.25, \]  
(17)

In addition to that, we have to drop the second summand in Eq. (13) since it is in a conflict with the approximation used for derivation of Eq. (9) (see sub-section 2*). Therefore, instead of Eq. (13) we now have
\[ x = C_1 t^{\frac{1}{2} + s} \]  
(18)

For illustration, let us evaluate the constant based upon the following data:
\[ \hbar = 10^{-34} m^2 kg/sec, m = 10^{-30} kg, L = 2.8 \times 10^{-15} m, L/T = \bar{C} = 3 \times 10 m/sec \]

where \( m \) - mass of electron, and \( \bar{C} \) - speed of light. Then,
\[ c = -1.5 \times 10^{-4}, \] i.e. \( |c| < 0.25 \)

Hence, the value of \( c \) is within the limit (17). Thus, for the particular case under consideration, the solution (18) is
\[ x = C_1 t^{0.9998} \]  
(19)

In the next sub-section, prior to analysis of the solution (18), we will present the proofs justifying the solution (10).

2*. Proofs.

1. Let us first justify the statement that \( \xi \to 0 \) at \( t \to 0 \) (see Eq. (8).

For that purpose, consider the solution (10)
\[ \rho = \frac{1}{\sqrt{4\pi at}} \cos\left(\frac{X^2}{4} - \frac{\pi}{4}\right) \]  
(1*)

As follows from the solution (18),
\[ \frac{x}{t} \approx o(t^{s-1/2}) \rightarrow \infty, \quad \frac{x^2}{t} \approx o(t^{2s}) \rightarrow 0 \text{ at } t \to 0 \text{ since } 0 < s < 1/2 \]  
(2*)

Then, finding the derivatives from Eq. (1') yields
\[ \left| \frac{\partial^n \rho}{\partial X^n} \right| \left| \frac{\partial^{n-1} \rho}{\partial X^{n-1}} \right| \approx o(t^{-1}) \rightarrow \infty \text{ at } t \to 0 \]  
(3*)
and that justifies the inequalities
\[ \left| \frac{\partial^4 \rho}{\partial X^4} \right| \gg \left| \frac{\partial^2 \rho}{\partial X^2} \right|, \left| \frac{\partial^2 \rho}{\partial X} \right|, \rho \quad (4^*) \]

Similarly,
\[ \left| \frac{\partial^n \rho}{\partial t^n} \right| / \left| \frac{\partial^{n-1} \rho}{\partial t^{n-1}} \right| \approx o(t^{-1}) \to \infty \text{ at } t \to 0 \quad (5^*) \]

and that justifies the inequalities
\[ \left| \frac{\partial^2 \rho}{\partial t^2} \right| \gg \left| \frac{\partial \rho}{\partial t} \right|, \left| \frac{\partial \rho}{\partial X} \right|^2 \]

Also as follows from the solution (18)
\[ \left| \frac{\partial S}{\partial x} \right| \approx o(t^{S-0.5}), \left| \frac{\partial^2 S}{\partial x^2} \right| \approx o(t^{-1}), \left| \frac{\partial S}{\partial t} \right| / \left| \frac{\partial S}{\partial x} \right| \approx o(t^{S+0.5}) \to 0 \text{ at } t \to 0 \quad (6^*) \]

It should be noticed that for Eq. (13), the evaluations (6*) do not go through, and that was the reason for dropping the second summand.

Finally, the inequalities (4*), (5*) and (6*) justify the transition from Eq. (8) to Eq. (10).

2. Next let us first prove the positivity of \( \rho \) in Eq. (10) for small times. Turning to the evaluation (2*)
\[ \frac{x^2}{t} \approx o(t^{2^*}) \to 0 \text{ at } t \to 0, \text{ one obtains for small times} \]
\[ \rho = \frac{1}{\sqrt{4 \pi at}} \cos\left(\frac{-\pi}{4}\right) > 0 \text{ at } t \to 0 \quad (7^*) \]

In order to prove that \( \rho \) is normalized for small times, turn to Eq.(9) and integrate it over \( X \)
\[ \int_{-\infty}^{\infty} \frac{\partial^2 \rho}{\partial t^2} dX + a^2 \int_{-\infty}^{\infty} \frac{\partial^4 \rho}{\partial X^4} dX = 0 \quad (8^*) \]

Taking into account the initial conditions (7) and requiring that \( \rho \) and all its space derivatives vanish at infinity, one obtains
\[ \frac{\partial^2}{\partial t^2} \int_{-\infty}^{\infty} \rho dX = 0 \quad (9^*) \]

But as follows from the initial conditions (7)
\[ \int_{-\infty}^{\infty} \rho dX = 0, \frac{\partial}{\partial t} \int_{-\infty}^{\infty} \rho dX = 0 \text{ at } t = 0 \quad (10^*) \]

Combining Eqs. (9*) and (10*), one concludes that the normalization constraint is preserved during small times.
3. The solutions (13), (14) and (15) have been derived under assumption that

\[ \frac{x^2}{t} \to 0 \text{ at } t \to 0 \]  

(20)

since this assumption was exploited for expansion of \( \rho \) in Eq. (10) in Taylor series. However, in the cases (14) and (15),

\[ \frac{x^2}{t} \approx o(1) \text{ at } t \to 0, \]

and that disqualify their derivation. Actually these cases require an additional analysis that is out of scope of this paper. For the same reason, Eq. (13) has been truncated to the form (18).

3. Analysis of solution

Turning to the solution (18), we notice that it satisfies the initial condition (7) i.e. \( x = 0 \) at \( t = 0 \) for any values of \( C_1 \): all these solutions co-exist in a superimposed fashion; it is also consistent with the sharp initial condition for the solution (10) of the corresponding Liouville equation (1). The solution (10) describes the simplest irreversible motion: it is characterized by the “beginning of time” where all the trajectories intersect (that results from the violation of the Lipchitz condition at \( t = 0 \), Fig.2); then the solution splits into a continuous set of random samples representing a stochastic process with the probability density \( \rho \) controlled by Eq. (10). The irreversibility of the process follows from the fact that the backward motion obtained by replacement of \( t \) with \((-t)\) in Eqs. (10) and (18) leads to imaginary values. Actually Fig. 1 illustrates a jump from determinism to a coherent state of superimposed solutions that is lost in solutions of the Schrödinger equation.

![Fig. 1 Hidden statistics of transition from determinism to randomness.](image)

Let us show that this jump is triggered by instability of the deterministic state. Indeed, turning to the solution represented by Eq. (18) with \(|C_1| \leq 0.25\), we observe that
for fixed values of $C_1$, the solution (18) is **unstable** since

$$\frac{d\dot{x}}{dx} = \frac{\ddot{x}}{\dot{x}} > 0$$

and therefore, an initial error always grows generating **randomness**. Initially, at $t = 0$, that growth is of **infinite rate** since the Lipchitz condition at this point is violated (such a point represents a **terminal** repeller.)

$$\frac{d\dot{x}}{dx} \to \infty \text{ at } t \to 0$$

This means that an **infinitesimal** initial error becomes finite in a bounded time interval. That kind of instability (similar to blow-up, or Hadamard, instability) has been analyzed in [5]. Considering first Eq.(18) at fixed $C_1$ as a sample of the underlying stochastic process (13), and then varying $C_1$, one arrives at the whole ensemble of one-parametrical random solutions characterizing that process, (see Fig.2). It should be stressed again that this solution is valid only during a small initial period representing a “bridge” between deterministic and random states, and that was essential for the derivation of the solutions (18), and (10).

![Fig. 2 Family of random trajectories and particle velocities.](image)

Returning to the quantum interpretation of Eqs. (1) and (2), one notice that during this transitional period, the quantum postulates are preserved. Indeed, as follows from Eq. (21),

$$\dot{x} \to \infty \text{ at } t \to 0$$

i.e. the initial velocity is not defined, (see the yellow areas in Fig. 2), and **that confirms the uncertainty principle**. It is interesting to note that an enforcement of the initial velocity would “blow-up” the solution (18); at the same time, the qualitative picture of the solution is not changed if the initial velocity is not enforced: the solution is composed of superposition of a family of random trajectories with the singularity (23) at the origin. Next, the solution (18) justifies the belief sheared by the most physicists that particle trajectories do not exist, although, to be more precise, as follows from Eq. (18), **deterministic** trajectories do not exist: each run of the solution (18) produces different trajectory that occurs with probability governed by Eq. (10). It is easily verifiable that the transition of motion from one trajectory to another is very sensitive to errors in initial conditions in the neighborhood of the deterministic state. Indeed, as follows from Eq. (18),

$$C_1 = x_0 t_0^{-(s+0.5)}, \quad \frac{\partial C_1}{\partial x_0} = t_0^{-(s+0.5)} \to \infty \text{ as } t_0 \to 0$$

(24)
where $x_0$ and $t_0$ are small errors in initial conditions.

Actually Eq. (18) represents a hidden statistics of the underlying Schrödinger equation. As pointed out above, the cause of the randomness is non-Lipchitz instability of Eq. (18) at $t=0$. Therefore, trajectories of quantum particles have the same “status” as trajectories of classical particles in a turbulent or chaotic motion with the only difference that the “choice” of the trajectory is made only at $t_0 \rightarrow 0$. It should be emphasized again that the transition (18) is irreversible. However, as soon as the difference between the current probability density and its initial sharp value becomes finite, one arrives at the conventional quantum formalism described by the Schrödinger, as well as the Madelung equations. Thus, in the conventional quantum formalism, the transition from the classical to the quantum state has been lost, and that created a major obstacle to interpretation of quantum mechanics as an extension of the Newtonian mechanics. However, as demonstrated above, the quantum and classical worlds can be reconciled via the more subtle mathematical treatment of the same equations. This result is generalizable to multidimensional case as well as to case with external potentials.

4. Comments on equivalence of Schrödinger and Madelung equations

Equivalence of Schrödinger and Madelung equations was questioned by some quantum physicists on the ground that to recover the Schrödinger equation from the Madelung equation, one must add by hand a quantization condition, as in the old quantum theory. However, this argument has been challenged by other physicists. We will not go into details of this discussion since we will be more interested in mathematical rather than physical equivalence of Schrödinger and Madelung equations. Firstly we have to notice that the Schrödinger equation is more attractive for computations due to its linearity, while the Madelung equations have a methodological advantage: they allow one to trace down the Newtonian origin of the quantum physics. Indeed, if one drops the Planck’s constant, the Madelung equations degenerate into the Hamilton-Jacobi equation supplemented by the Liouville equation. However despite the fact that these two forms of the same governing equations of quantum physics can be obtained from one another (see Eqs. (3) and (5)) without a violation of any of mathematical rules, there is more significant difference between them, and this difference is associated with the concept of stability.

4.1 Stability in Physics

Any mathematical model of a continuum should be tested for three properties: existence, uniqueness and stability of its solutions. However, none of these properties are physical invariants since they depend upon a mathematical setting of the corresponding model. As an example, consider a vertical, ideally flexible filament OA with a free lower end A suspended in the gravity field at the point O, Fig. 3.

As shown in textbooks on analytical mechanics, [6], the problem of small oscillations
The tension $T$ of the filament due to gravity is the following

$$T = \gamma(L - x)$$

where $\gamma$ and $L$ are the specific weight and length of the filament.

4.1.0.1 Since the characteristic speed $\lambda$ of a transverse wave in ideal filaments is

$$\lambda = \sqrt{\frac{T}{\rho}}$$

this speed vanishes at the free end

$$T|_{x=L} = 0, \quad \lambda = 0 \text{ at } x \to 0$$

In other words, for small transverse displacements of the filament, the governing equation is of hyperbolic type only in the open interval that excludes the free end

$$0 \leq x < L$$

As shows in [7], in this open interval there exists a unique stable solution.

However, in the closed interval that includes the free end

$$0 \leq x \leq L$$

the solution is not unique, and there are unstable solutions since the improper integral

$$\int_0^x \frac{d\xi}{\sqrt{T(\xi)/\rho}}$$

converges for $x \to L$.

This result has a clear physical interpretation: suppose that an isolated transverse wave of small amplitude was generated at the point of suspension O, Fig. 3. Then the
speed of propagation of its leading front will be smaller than the speed of propagation of
the trailing front because the tension decreases from the point of suspension to the free
end (see Eq. (25))). Hence the length of the wave will decrease and vanish at the free
end. Then according to the law of conservation of energy, the kinetic energy per unit
of length will tend to infinity and produce a snap. It can be verified that the Lipchitz
condition at the free end is violated
\[
\frac{dx}{dx} \to \infty \text{ at } x \to L
\]
in the closed interval (29).

Thus it turns out that the unique stable solution exists in the class of functions
satisfying the Lipchitz condition. However despite its “nice” mathematical properties,
this solution is in contradiction with experiments: the cumulative effect – snap of a whip
– is lost. At the same time, the removal of the Lipchitz conditions leads to non-unique
unstable solutions that perfectly describe the snap of a whip.

This trivial example leads to an important conclusion: existence, uniqueness and sta-
bility of solutions of PDE describing dynamics of a continuum are not physical invariants:
they are attributes of underlying mathematical model. It also becomes clear that in
some cases, unstable and non-unique solutions are closer to reality than a unique and
stable one.

4.2 Euler/Lagrange description of fluid

In this subsection, we prepare a reader to analogy between the Schrödinger/Madelung
description of quantum mechanics and the Euler/Lagrange description of fluid.

The Lagrange’s description of fluid has advantage over the Euler’s one in case of studies
of mixing and dispersion. That includes mixing of passive scalars and the dispersion
of contaminants.

The Lagrange’s description of motion starts with the frame of reference that is frozen
at the fluid and moves with it. Initially, at \( t_0 = 0 \), such frame can be represented by
Cartesian axes \( X_0, Y_0, \) and \( Z_0 \).

For \( t > 0 \), these axes, in general, transform into a non-orthogonal curvilinear system.
Any individualized particle of the fluid with the coordinates \( x_0, y_0, \) and \( z_0 \) will have the
same coordinates in the mowing frame, but different coordinates in the initial frame
\[
x = x(x_0, y_0, z_0, t)
\]
\[
y = y(x_0, y_0, z_0, t)
\]
\[
z = z(x_0, y_0, z_0, t)
\]
Eqs. (32), (33) and (34) represent the Lagrange’s description of a continuum, including
fluid. These equations can be obtained as the solution of the governing equations of the
fluid in Euler’s description
\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} = F_x - \frac{1}{\rho} \frac{\partial p}{\partial x}
\]
\[ \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} = F_y - \frac{1}{\rho} \frac{\partial p}{\partial x} \]  
\( (36) \)

\[ \frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} = F_z - \frac{1}{\rho} \frac{\partial p}{\partial x} \]  
\( (37) \)

\[ \frac{\partial}{\partial x} u + \frac{\partial}{\partial y} v + \frac{\partial}{\partial z} w = 0, \]  
\( (38) \)

Presented in the Lagrange’s form

\[ (F_x - \frac{\partial^2 x}{\partial t^2}) \frac{\partial x}{\partial x_0} + (F_y - \frac{\partial^2 y}{\partial t^2}) \frac{\partial y}{\partial y_0} + (F_z - \frac{\partial^2 z}{\partial t^2}) \frac{\partial z}{\partial z_0} = \frac{1}{\rho} \frac{\partial p}{\partial x_0} \]  
\( (39) \)

\[ (F_x - \frac{\partial^2 x}{\partial t^2}) \frac{\partial x}{\partial y_0} + (F_y - \frac{\partial^2 y}{\partial t^2}) \frac{\partial y}{\partial y_0} + (F_z - \frac{\partial^2 z}{\partial t^2}) \frac{\partial z}{\partial y_0} = \frac{1}{\rho} \frac{\partial p}{\partial y_0} \]  
\( (40) \)

\[ (F_x - \frac{\partial^2 x}{\partial t^2}) \frac{\partial x}{\partial z_0} + (F_y - \frac{\partial^2 y}{\partial t^2}) \frac{\partial y}{\partial z_0} + (F_z - \frac{\partial^2 z}{\partial t^2}) \frac{\partial z}{\partial z_0} = \frac{1}{\rho} \frac{\partial p}{\partial z_0} \]  
\( (41) \)

\[ \begin{vmatrix} \frac{\partial x}{\partial x_0} & \frac{\partial y}{\partial x_0} & \frac{\partial z}{\partial x_0} \\ \frac{\partial x}{\partial y_0} & \frac{\partial y}{\partial y_0} & \frac{\partial z}{\partial y_0} \\ \frac{\partial x}{\partial z_0} & \frac{\partial y}{\partial z_0} & \frac{\partial z}{\partial z_0} \end{vmatrix} = 1 \]  
\( (42) \)

This is a system of four PDE with respect to four unknowns: \( x, y, z \) and \( p \) as functions of \( x_0, y_0, z_0 \) and \( t \).

Here \( p \) and \( F \) are pressure and force, respectively.

However if the Euler’s equations (35) – (38) are already solved in the form

\[ u = u(x, y, z, t) \]  
\[ v = v(x, y, z, t) \]  
\[ w = w(x, y, z, t) \]  
\( (43) \)

then the Lagrangian description of the same motion reduces to three kinematical ODE

\[ \frac{dx}{dt} = u(x, y, z, t) \]  
\[ \frac{dy}{dt} = v(x, y, z, t) \]  
\[ \frac{dz}{dt} = w(x, y, z, t) \]  
\( (44) \)

To be solved subject to the following initial conditions

\[ x = x_0, \quad y = y_0, \quad z = z_0 \text{ at } t = 0 \]  
\( (45) \)

Or in the vector form

\[ \frac{d\mathbf{r}}{dt} = \mathbf{v}(\mathbf{r}, t) \]  
\( (46) \)

The solution of this system can be written in the form (32) – (34).
Actually Eq. (46) is the analog of Eq. (3): it describes the transition from the Euler’s to Lagrange’s description of fluid in the same way as Eq. (3) describes the transition from the Schrödinger’s to Madelung description of quantum mechanics. The inverse transition similar to Eq. (5) is

\[ \mathbf{v}_L(r_0, t) = \mathbf{v}_E[r_0(r, t), t] \]  

(see Eqs. (32-34))

For further analysis we will restrict the problem to stationary laminar flows and consider the autonomous version of Eqs. (44)

\[ \frac{dx}{dt} = u(x, y, z) \]  
\[ \frac{dy}{dt} = v(x, y, z) \]  
\[ \frac{dz}{dt} = w(x, y, z) \]

The first theoretical example of the case when the flow in the Euler’s description is stable, but in the Lagrange’s description is unstable was introduced by V. Arnold, [8].

He considered a 3D inviscid stationary shear flow with a smooth velocity field

\[ u = A \sin z + C \cos y, \quad v = B \sin x + A \cos z, \quad w = C \sin y + B \cos x \]  

(see Fig. 4)

![Cartesian components of the velocity vector.](image)

The solution (51) is differentiable as many times as needed, it satisfies the Euler equations (35)-(38), and it is stable. However if one tries to find the trajectories of individual particles by transition to the Lagrangian description, i.e.to Eqs. (48)-(50), he finds that after substitution the solution (51), these equations

\[ \frac{dx}{dt} = A \sin z + C \cos y \]  
\[ \frac{dy}{dt} = B \sin x + A \cos z \]  
\[ \frac{dz}{dt} = C \sin y + B \cos x \]

are unstable, and their solution is chaotic. This means that a flow with stable stationary deterministic velocity field can have non-stationary random trajectories of individual particles. In other words, it means that this flow is stable in the Eulerian coordinates,
but is unstable in the Lagrangian coordinates. Experimentally this phenomenon can be captured if some particles of the flow are marked by different colors. This interesting and surprising phenomenon opened up a new direction in fluid mechanics known as Lagrangian turbulence.

4.3 Schrödinger/Madelung and Euler/Lagrange analogy

Actually the result presented above demonstrates the analogy between quantum mechanics and fluid mechanics. Indeed, as demonstrated in Sections 2 and 3 of this paper, the solution of the Madelung equations with deterministic initial condition (7) is unstable, and it describes the jump from the determinism to randomness. This illuminates the origin of randomness in quantum physics. However the Schrödinger equation does not have such a solution; moreover, it does not “allow” one to pose such a problem and that is why the randomness in quantum mechanics had to be postulated. So what happens with mathematical equivalence of the Schrödinger and the Madelung equations? In order to answer this question, let us turn again to the concept of stability. It should be recalled that stability is not an invariant of a physical model. It is an attribute of its mathematical description: it depends upon the frame of reference, upon the class of functions in which the motion is presented, upon the metrics of configuration space, and in particular, upon the way in which the distance between the basic and perturbed solutions is defined. One should recall that stability analysis is based upon a departure from the basic state into a perturbed state, and such departure requires an expansion of the basic space. However, Schrödinger and Madelung equations in the expanded spaces are not necessarily equivalent any more, and that explains the difference in the concept of stability of the same solution as well as the interpretation of randomness in quantum mechanics.

There is another “mystery” in quantum mechanics that can be clarified by transition to the Madelung space: a belief that a particle trajectory does not exist. Indeed, let us turn to Eq. (18). For any particular value of the arbitrary constant $C_1$, it presents the corresponding particle’s trajectory. However as a result of the Lipchitz instability at $t = 0$, this constant is supersensitive to infinitesimal disturbances, and actually it becomes random at $t = 0$. That makes random the choice of the whole trajectory, while the randomness is controlled by Eq. (1). Actually this provides a justification for the belief that a particle can occupy any place at any time: it is due to randomness of its trajectory. However it should be emphasized that the particle makes random choice only once:

at $t = 0$. After that it stays on the chosen trajectory. Therefore in our interpretation this belief does not mean that a trajectory does not exist: it means only that the trajectory exists, but it is unstable. Based upon that, we can extract some deterministic information about the particle trajectory by posing the following question: find such a trajectory that has the highest probability to appear. The solution of this problem is straight forward: in the process of collecting statistics for the arbitrary constant $C_1$ find such its value that has the highest frequency to appear. Then the corresponding trajectory will have the
highest probability to appear as well.

Thus, for the problem with the initial condition (7), the Schrödinger and Madelung equations are equivalent only in the open time interval

$$t > 0,$$  \hfill (55)

since the Schrödinger equation does not include the infinitesimal area around the singularity at

$$t = 0$$  \hfill (56)

while the Madelung equation exists in the closed interval

$$t \geq 0$$  \hfill (57)

But all the “machinery” of randomness emerges precisely in the area around the singularity (56). That is why the source of randomness is missed in the Schrödinger equation, and the randomness had to be postulated.

Hence although historically the Schrödinger equation was proposed first, and only after a couple of months Madelung introduced its hydrodynamic version that bears his name, strictly speaking, the foundations of quantum mechanics would be saved of many paradoxes had it be based upon the Madelung equation.

5. Summary

Reformulation of quantum mechanics using the Madelung equation allows one to clarify the origin of randomness and justify the belief that a particle can occupy any position at any time. The clarifications are based upon the blow-up instability of a deterministic state due to failure of Lipchitz condition. This property does not exist in Hilbert space formulation. It has been demonstrated that randomness in quantum mechanics has the same mathematical source as that in turbulence and chaos. Special attention is concentrated on equivalence between the Schrodinger and the Madelung equations in connection with the concept of stability in dynamics.

References
