The Hadronic Decay Ratios of $\eta' \to 5\pi$ at NLO in $\chi$PT

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Abstract: The hadronic decays $\eta'(958) \to 2\pi\eta$ and $\eta \to 3\pi$ give unique scientific opportunity to study symmetries in nature and to provide experimental verification of the sophisticated theoretical predictions and to reveal the driving mechanism of the decays as well as violating of isospin symmetry. We give the next-to-leading order (NLO) results in Chiral Perturbation Theory ($\chi$PT) for the ratio of different $\eta'(958) \to 5\pi$ decay rates. The hadronic decay modes are also discussed and in particular some interesting features of the $\eta'$ decay are presented. The scenario we have considered shows reasonable agreement with the decay processes observed so far.

Keywords: Chiral Perturbation Theory; Hadronic Decay; Quark Models; Mesons


1. Introduction

When the $\eta'$ discovered in 1964, there has been considerable interest in its decay both theoretically and experimentally because of its special role in low-energy scale Quantum Chromodynamics (QCD) theory. Its main decay modes, including hadronic and radiative decays, have been well measured but study of its anomalous decays is still an open field. Study of $\eta'$ decays provides important tests of Chiral Perturbation Theory ($\chi$PT) and other models of strong interactions, allows measurement of the EM transition form factors and tests of discrete symmetries such as C, P, CP and physics beyond the Standard model. An appropriate theoretical framework to investigate hadronic physics is provided by model-independent calculations $\chi$PT that can be applied for strong interactions in hadronic reactions and decays[1-11]. $\chi$PT, the low energy effective theory of QCD with $N_f$ light quark flavors, could posses different behavior for $N_f=2$ and 3. At small distances the

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gauge coupling decreases logarithmically, and the dynamics is successfully described by $\chi$PT close to threshold, however, for the energy range where only the massless Goldstone bosons participate [12].

$\eta'$ has been the center of attention in many theoretical and experimental works over the recent decades. The inclusion of the $\eta'$ spoils the conventional chiral counting scheme, since its mass does not vanish in the chiral limit so that higher loops will still contribute to lower chiral orders. The $\eta'$ is a particle that decay strongly but all its decays is suppressed. $\eta'$ decays offers an introduction to some of the basic strong interaction issues to study symmetries and symmetry breaking patterns.

There are many study by examining hadron decays $\eta' \rightarrow \pi\pi\eta$ [1-4]. The energy dependence of hadronic $\eta' \rightarrow \pi\pi\eta$ decay is discussed in the $\chi$PT framework in Ref. [4]. The $\eta'$ can be decayed (through both of its charged and neutral decay channels) into $5\pi$. These pions will not be correlated in terms of Bose-Einstein correlations. Preliminary results strongly support the mass decrease of the $\eta'$ boson [13-15]. The $\eta' \rightarrow 3\pi$ decays do not conserve isospin since Bose symmetry forbids the three pions with $J^P = 0^-$ to occur in the iso-scalar state. Due to the large mass of $\eta'$ light vector and scalar mesons could be produced in the decays. Importance of vector mesons is seen in radiative decay modes; $\eta' \rightarrow \rho\gamma$ and $\eta' \rightarrow \omega\gamma$. Contributions of light scalar mesons $\sigma(560)$, $f_0(980)$ and $a_0(980)$ should play a significant role in the decays into $\eta\pi\pi$ and $\pi\pi\pi$ [12] but it is not as apparent due to the width of the mesons and the off-shell behavior. Description of vector and scalar resonances is beyond the scope of the standard $\chi$PT but there is continuous progress in the theoretical treatment.

In the present work, we calculate the $\eta'(958) \rightarrow 5\pi$ decay rates using $\chi$PT. Within the framework of $\chi$PT, electromagnetic correction can also be described by a series of effective operators of increasing power in momentum or masses of the mesons. This will also allow us to obtain more realistic predictions for the decay $\eta'$. Furthermore, we first present the results which we obtain by relying purely on the numbers quoted by the Particle Data Group (PDG) in different years [16, 17, 18].

The paper is organized as follows. In next section, we give a brief description of theoretical framework of this work. The $\chi$PT Lagrangian, definition of the different levels of Lagrangian, the scattering amplitudes and decay rates and other parameters are represented in this section. We are devoted to comparison of our results with the corresponding other experimental and theoretical results, in section III. Summary and conclusions follow in Section IV.

2. Theoretical framework

We introduced the relevant chiral Lagrangian for the calculation of octet meson masses in the three-flavor sector of $u, d$ and $s$ quarks, up to NLO. In this formalism, the most general chiral Lagrangian is Lorentz invariant, symmetric under chiral transformation and also contains the pseudo-scalar mesons. The effective Lagrangian consists of operators ,
Fig. 1 The tree level diagrams contributing to $\eta' \to 5\pi$. In this figure two successive decay occurs: filled circle comes from $L_2$, solid and dashed lines denote input or output pions and $\eta$ particle, respectively.

with a definite number of derivatives, is given by

$$\mathcal{L}_{\text{eff}} = \mathcal{L}_2 + \mathcal{L}_4 + \ldots,$$

(1)

where $\mathcal{L}_2$, the leading order part of the Lagrangian, has the form

$$\mathcal{L}_2 = \frac{F_0^2}{4} \left( \text{Tr}(D_\mu U^\dagger D^\mu U) + \text{Tr}(\chi U^\dagger U + \chi U^\dagger) \right),$$

(2)

and $D_\mu U = \partial_\mu U + \{A_\mu, U\} + [V_\mu, U]$, where $A_\mu$ and $V_\mu$ are the axial and vector currents, respectively. The $\mathcal{L}_2$ term of lagrangian contains the $SU(3)$ chiral limit of the Goldstone boson decay constant $F_0$ and the external pseudo-scalar sources that are conventionally combined into $\chi = 2B_0M$ and accounts for explicit chiral symmetry breaking. $B_0$ is equal to $-\frac{\langle 0 | q q | 0 \rangle}{\langle 0 | q q | 0 \rangle}$ and is related to the chiral quark condensate with $M = \text{diag}(m_u, m_d, m_s)$. The $U$ matrix, including of the Goldstone boson fields and singlet meson, has the form

$$U = \exp(i \frac{\phi}{F_0}), \quad \phi = \begin{pmatrix} \pi^0 + \frac{1}{\sqrt{3}} \eta_8 + \frac{\sqrt{2}}{\sqrt{3}} \eta_0 & \sqrt{2} \pi^+ & \sqrt{2} K^+ \\ \sqrt{2} \pi^- & \pi^0 + \frac{1}{\sqrt{3}} \eta_8 + \frac{\sqrt{2}}{\sqrt{3}} \eta_0 & \sqrt{2} K^0 \\ \sqrt{2} K^- & \sqrt{2} K^0 & \frac{2}{\sqrt{3}} \eta_8 + \frac{\sqrt{2}}{\sqrt{3}} \eta_0 \end{pmatrix}.$$  

(3)

where, the $\eta_8, \eta_0$ are related to the mass eigenstates of $\eta, \eta'$ [1]

$$\eta_8 = \epsilon \pi^0 + \eta - v \eta', \quad \eta_0 = 3 \epsilon \nu \pi^0 + v \eta + \eta'.$$

(4)

and $\epsilon$ and $v$ are dependent to the difference mass of $u, d$ and $s$ quarks( for more details see Ref. [1]).

The next-to-leading order($O(p^4)$) part of the Lagrangian is given by

$$\mathcal{L}_4 = L_1 \text{Tr}[D_\mu UD^\mu U^\dagger]^2 + L_2 \text{Tr}[D_\mu UD_\nu U^\dagger] \text{Tr}[D^\mu U D^\nu U^\dagger]$$

$$+ L_3 \text{Tr}[D_\mu UD^\mu U^\dagger] D_\nu U D^\nu U^\dagger] + L_4 \text{Tr}[D^\mu U D^\nu U^\dagger] \text{Tr}(\chi U^\dagger + U \chi^\dagger)$$

$$+ L_5 \text{Tr}[D_\mu UD^\mu U^\dagger(\chi U^\dagger + U \chi^\dagger)] + L_6 \text{Tr}(\chi U^\dagger + U \chi^\dagger)^2$$

$$+ L_7 \text{Tr}(\chi U^\dagger - U \chi^\dagger)^2 + L_8 \text{Tr}(\chi U^\dagger U^\dagger + U \chi^\dagger U^\dagger)$$

$$- i L_9 \text{Tr}[f_{\mu \nu} D^\mu UD^\nu U^\dagger + f_{\mu \nu} D^\mu U D^\nu U^\dagger] + L_{10} \text{Tr}[U f_{\mu \nu} U^\dagger f^{R \mu \nu}].$$

(5)
where \( f^R_{\mu}, f^L_{\mu} \) are external field strength tensors defined via
\[
f^R_{\mu \nu} = \partial_\mu f^R_{\nu} - \partial_\nu f^R_{\mu} - i [f^R_{\mu}, f^R_{\nu}], \quad f^L_{\mu} = V_\mu \pm A_\mu.
\]
(6)

It is noted that, the last line of the \( \mathcal{L}_4 \) lagrangian only contains the external fields and is not significant for low-energy physics.

The coupling constants \( L_i (i = 0, \ldots, 12) \) are not specified from chiral symmetry. \( L_i \)'s are measurable quantities and can be determined phenomenological. The relevant experiment of these parameters are obtained by appending to these values the divergent one-loop contributions
\[
L_i = L_i^r + \frac{\Gamma_i}{32\pi^2} R, \quad R = \frac{2}{n - 4} - [\ln(4\pi) - \gamma_E + 1],
\]
(7)
where \( \gamma_E = 0.5772 \ldots \) is Euler’s constant and \( n \) implication the number of space-time dimensions. The renormalization of amplitude at this order can be done when the \( \Gamma_i \)'s are found in such a way that the divergences of the one loop Green functions divergences of the tree level diagrams of \( \mathcal{L}_4 \) vanished. Gasser and Leutwyler applied the background field method for calculation of the divergences at NLO [15].

The basic features of the \( \eta' \) meson from the most recent issue of the Particle Data Group report [18] are summarized in table 1.

Here, we explain how to compute the decay amplitudes for both the charged and the neutral channel of \( \eta' \) decays at next-to-leading chiral order \( p^4 \) considering isospin breaking up to. When the \( \eta' \) goes into an \( \eta \) and \( 2\pi \), and then \( \eta \) decays into \( 3\pi (\eta' \rightarrow 2\pi \eta \rightarrow 5\pi) \), there are four different CP-conserving decays
\[
\begin{align*}
\eta'(k) & \rightarrow \pi^0(p_1) + \pi^0(p_2) + \eta(p_3) \rightarrow \pi^0(p_1) + \pi^0(p_2) + \pi^0(p_4) + \pi^0(p_5) + \pi^0(p_6), & [A_{00000}], \\
\eta'(k) & \rightarrow \pi^+(p_1) + \pi^-(p_2) + \eta(p_3) \rightarrow \pi^+(p_1) + \pi^-(p_2) + \pi^+(p_4) + \pi^-(p_5) + \pi^0(p_6), & [A_{+++0}] \\
\eta'(k) & \rightarrow \pi^+(p_1) + \pi^-(p_2) + \eta(p_3) \rightarrow \pi^+(p_1) + \pi^-(p_2) + \pi^0(p_4) + \pi^0(p_5) + \pi^0(p_6), & [A_{+000}] \\
\eta'(k) & \rightarrow \pi^0(p_1) + \pi^0(p_2) + \eta(p_3) \rightarrow \pi^0(p_1) + \pi^0(p_2) + \pi^+(p_4) + \pi^-(p_5) + \pi^0(p_6), & [A_{00+0}].
\end{align*}
\]
(8)
Table 1 Main properties of the $\eta'$ meson [18].

<table>
<thead>
<tr>
<th>$M_{\eta'}$</th>
<th>$\Gamma_{\eta'}$</th>
<th>Fit</th>
</tr>
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<tbody>
<tr>
<td>$957.793 \pm 0.06$ MeV</td>
<td>$0.230 \pm 0.021$ MeV</td>
<td>$0.199 \pm 0.009$ MeV</td>
</tr>
</tbody>
</table>

$\eta' \to \pi^+\pi^-\eta$ \hspace{1cm} (43.4 \pm 0.7)%
$\eta' \to \rho^0\gamma$ \hspace{1cm} (29.3 \pm 0.6)%
$\eta' \to \pi^0\pi^0\eta$ \hspace{1cm} (21.6 \pm 0.8)%
$\eta' \to \omega\gamma$ \hspace{1cm} (2.75\pm0.22)%
$\eta' \to \gamma\gamma$ \hspace{1cm} (2.18\pm0.08)%
$\eta' \to \pi^+\pi^-\pi^0$ \hspace{1cm} $(3.6\pm1.0)\times10^{-3}$%

Where the $p$ and $A$ labels are indicated the four-momentum defined for each particle and the symbol used for the each amplitude, respectively. The neutral decay is described at leading order by the diagram shown in fig. 1. The lowest-order amplitude contains neither derivatives nor electromagnetic terms. The neutral LO amplitude also has an overall factor of $m_d - m_u$, but is just a constant. Fig. 2 is also shown the contributing Feynman diagrams up to next-to-leading chiral order $p^4$ for $\eta' \to 5\pi$ decay.

The kinematics is also normally treated by using

\[
\begin{align*}
    s_1 &= (k - p_1)^2, \quad s_2 = (k - p_2)^2, \quad s_3 = (k - p_3)^2, \\
    s'_1 &= (p_3 - p_4)^2, \quad s'_2 = (p_3 - p_5)^2, \quad s'_3 = (p_3 - p_6)^2.
\end{align*}
\]

(9)

where $p_1^2 = m_i^2$ and $k^2 = m_{\eta'}^2$. The $A_{00+00}$, $A_{+0-00}$ and $A_{+-+0}$ amplitudes are symmetric under the interchange of the first and second two pions, because of CP or Bose-symmetry. The $[A_{0000}]$ amplitude is also obviously symmetric under the interchange of all the final state particles.

In terms of the contributions defined before and by applying the replacement rules given above the decay amplitudes finally can be written in terms of single variable functions $M_i(s)$ and $M'_i(s)$ (see for more details Ref. [19])

\[
\begin{align*}
    A_{00000}(s_1, s_2, s_3) &= M_0(s_1) + M_0(s_2) + M_0'(s_3), \\
    A_{+-+00}(s_1, s_2, s_3) &= M'_1(s_3) + M_2(s_1) + M_2(s_2) + (s_2 - s_3)M_3(s_1) + (s_1 - s_3)M_3(s_2), \\
    A_{+0-00}(s_1, s_2, s_3) &= M'_1(s_3) + M_5(s_2) + M_5(s_2) + (s_2 - s_3)M_6(s_1) + (s_1 - s_3)M_6(s_2), \\
    A_{00+0-0}(s_1, s_2, s_3) &= M'_2(s_3) + M_8(s_1) + M_8(s_2) + (s_2 - s_3)M_9(s_1) + (s_1 - s_3)M_9(s_2).
\end{align*}
\]

(10)

The functions $M_i(s)$ and $M'_i(s)$ are not unique. Note that in case of the neutral decay the whole kinematical dependence is contained in the scattering contribution, which takes the
symmetric form where discussed in the above paragraph. We have also checked both the charged and the neutral amplitude in several ways. They are finite and renormalization-scale independent.

The charged and neutral total decay widths can be calculated

$$\Gamma_{c(n)} = \frac{S_{c(n)}}{256 \pi^3 m_{\eta}^2} \int |A(s_i)|^2 ds_i . \tag{11}$$

where $S_{c(n)}$ is denote the symmetry factor of the charged(neutral)decay.

3. Results and discussion

We are focus to the ratio between scattering amplitudes and decay widths. Thus, we introduce some new parameters. These parameters are the ratio between scattering amplitudes in deferent decay modes

$$r_1 = \left| \frac{A_{\eta'\rightarrow e^+e^-\pi^0\pi^0}}{A_{\eta'\rightarrow e^-e^+\pi^0\pi^0}} \right|^2 , \quad r_2 = \left| \frac{A_{\eta'\rightarrow e^+e^-\pi^0\pi^0}}{A_{\eta'\rightarrow e^-e^+\pi^0\pi^0}} \right|^2 , \quad r_3 = \left| \frac{A_{\eta'\rightarrow e^+e^-\pi^0\pi^0}}{A_{\eta'\rightarrow e^-e^+\pi^0\pi^0}} \right|^2 , \quad r_4 = \left| \frac{A_{\eta'\rightarrow e^+e^-\pi^0\pi^0}}{A_{\eta'\rightarrow e^-e^+\pi^0\pi^0}} \right|^2 . \tag{12}$$

The ratio of the $r_1$ to $r_4$ is shown in fig. 3. By comparison of the two diagrams fig. 3(a) and fig. 3(c), one can see that the ratio of $\eta' \rightarrow \pi^+\pi^-\pi^-\pi^0$ is nearly about two times of scattering amplitude $\eta' \rightarrow \pi^0\pi^0\pi^0\pi^0$. On the other hand, according to eq. (12), we conclude that decay widths of $\eta' \rightarrow \pi^+\pi^-\pi^-\pi^0$ two times of decay widths of $\eta' \rightarrow \pi^0\pi^0\pi^0\pi^0$ which is in good agreement to the experimental results [16, 17].

In fig. 4(a), the squared amplitude ratio $r_1, r_2$ and $r_3, r_4$, are also compared. This comparison is shown that by increasing of $s_2$, $r_2$ is growing faster than $r_1$. For $s_2 = 0.46$, $r_1$ and $r_2$ are equal to each other which indicates that, by increasing of $s_2$, the scattering amplitude in the neutral channel, is reduced. By increasing $s_2$, $r_4$ is also decreased, but $r_3$ is increased and for $s_2 = 0.375$, $r_3$ and $r_4$ are equal to each other, this behavior is shown in fig. 4(b).

The calculated of decay widths ratios as a function as $s_2$ is shown in fig. 5. Table 1 is also show the theoretical and measurement estimate of the decay widths ratios, up

<table>
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<th>Table 2 Theoretical and measurement estimate of the decay widths ratios up to NLO.</th>
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<td>( \frac{\Gamma(\eta'\rightarrow\pi^+\pi^-\pi^-\pi^0)}{\Gamma(\eta'\rightarrow\pi^0\pi^0\pi^0\pi^0)} )</td>
</tr>
<tr>
<td>Ref [16]</td>
</tr>
<tr>
<td>Ref [17]</td>
</tr>
<tr>
<td>Ref [18]</td>
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<tr>
<td>This work</td>
</tr>
</tbody>
</table>
to NLO. The calculated values of the decay widths ratios and other experimental and theoretical results are compared in this table. The ratio between decay width of the charge the neutral channels, is found to be 1.72, in comparison with the its experimental value 1.90 [18]. Due to the isospin symmetric calculations of identical particles, in the isospin symmetry breaking limit, $m_u = m_d$, the decay width of charged channel $\Gamma(\eta' \to \pi^+\pi^-\pi^+\pi^-\pi^0)$ would be given by two times of the decay width of neutral channel $\Gamma(\eta' \to \pi^0\pi^0\pi^0\pi^0\pi^0)$.

4. Summary and Conclusion

Qualitatively, the scenario we have considered shows reasonable agreement with the decay processes observed so far. Further experimental input would be appreciated for the $\eta(958)$.

In this work, we have studied the hadronic $\eta'$ decays in the frameworks of Chiral
Fig. 4 Comparison of $\eta' \rightarrow 5\pi$ amplitudes ratio between $r_1, r_2$ and $r_3, r_4$, the horizontal axis in terms of $s_2$.

Perturbation Theory, at lowest and next-to-leading orders. We investigate the scattering amplitude $\eta'$ particle decays into five particles and the scattering amplitude between the different graphing modes shown. In the end, decay widths ratios was calculated between different modes, as well as decay widths ratios between the different graphing modes is shown. We have done the calculations to NLO in $\chi$PT in the isospin limit. This work is considered the isospin breaking, the calculated results have good agreement with experimental results. To obtain more accurate results, the calculations can be extended to higher order as well.

References

Fig. 5 Comparison of \( \eta' \rightarrow 5\pi \) decays width ratio between different modes mentioned in this article, the horizontal axis in terms of \( s_2 \).
