Bianchi Type-VI_0 Cosmological Models with Isotropic Dark Energy

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Abstract: We present spatially homogeneous and anisotropic Bianchi-type VI_0 cosmological models of the universe filled with dark energy in general theory of relativity. The Einstein's field equations have been solved exactly with the assumption that the shear scalar is proportional to the expansion scalar, which for some suitable choices of problem parameters, yield singular and non-singular dark energy models of the universe. The physical and kinematical behaviors of the models are discussed. We conclude that the universe models do not approach isotropy through the future evolution of the universe.

Keywords: Bianchi type-VI_0 space time; Dark energy; Deceleration Parameter
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1. Introduction

Modern cosmology is concerned with nothing less than a thorough understanding and explanation of past history, the present, and the future evolution of the universe. It conventionally involve Einstein's general theory of relativity for determination of the gravitational field in terms of geometrical relationships. There is a good observational evidence that at our cosmological epoch the universe is fairly homogeneous on the large scale and has been highly isotropic. The Friedmann-Robertson-Walker models are the simplest models of the expanding universe which are spatially homogeneous and isotropic where the source of the gravitational field is most naturally considered to be a perfect fluid whose matter density ρ and pressure satisfy a equation of state of the form

$$p = \omega\rho \tag{1}$$

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where ω is the equation of state (EoS) parameter, not necessarily constant. The three most common examples of the cosmological fluids with constant ω are the dust ($\omega = 0$), radiation ($\omega = \frac{1}{3}$) and vacuum energy ($\omega = -1$) that is mathematically equivalent to the cosmological constant Λ. Israelet and Rosen $[1-2]$ presented a singularity-free cosmological model for Robertson-Walker line-element using an equation of state in which the pressure varies continuously from $-p$ to its value during the radiation era ($p = \frac{\rho}{3}$). They have studied the evolution of the universe for pre-matter ($p = -\rho$), radiation-dominated period and the matter dominated period. They have also described the transition between these periods. Carvalho [3] considered homogeneous and isotropic cosmological models with zero curvature and presented a unified description of the early universe by taking EoS parameter as function of the scale factor. Fluids with $\omega < -\frac{1}{3}$ are usually considered in the context of dark energy (DE), since they give rise to accelerating expansion. Surveys of cosmologically distant SN Ia (Riess et al.[4]; Perlmutter et al.[5] have indicated the presence of unaccounted dark energy that opposes the self-attraction of matter and causes the expansion of the universe to accelerate. This acceleration is realized with negative pressure and positive energy density that violates the strong energy condition. Other possible forms of DE include quintessence ($\omega > -1$) (Steinhardt et al. [6], phanton ($\omega < -1$)(Caldwell [7] etc.).

The theoretical arguments suggest and observational data show that the universe was anisotropic at the early stage. Bianchi space I-IX play important roles in constructing models spatially homogeneous and anisotropic cosmologies for describing the early stages of evolution of the universe. Here we confine ourselves to models of Bianchi type-VI_0. There is a large literature concerning Bianchi type-VI_0 spaces which contain fluids satisfying specific equation of state.

Berman [8], Berman and Gomide [9] proposed a law of variation for Hubble parameter within the context of Robertson-Walker space-time in general relativity that yields constant deceleration (DP) parameter

$$q = -\frac{a\ddot{a}^2}{\dot{a}^2} \tag{2}$$

where a is scale factor. In Berman's law the deceleration parameter can get values $q \geq -1$. For accelerating expansion of the universe, we must have $-1 \leq q < 0$. Many authors have studied cosmological models using this law in the context of DE following the discovery of current acceleration of the universe. Abdussattar and Prajapati [10] obtained a class of non-singular bouncing FRW models by constraing the deceleration parameter in the presence of an interacting dark energy represented by a time varying cosmological constant. The models, being geometrically closed, initially accelerate for a certain period of time and decelerate thereafter and are also free from the entropy and cosmological constant problem. Recently, Saha and Yadav [11] presented a spatially homogeneous and anisotropic LRS Bianchi type-II dark energy model in general relativity. They have obtained exact solutions of Einstein's field equations which for some suitable choices of problem parameters yield time dependent EoS and DP parameters, representing a model which generate a transition of universe from early decelerating phase to present accelerating phase. They have also studied DE models with variable EoS parameter (Yadav and Saha, [12]).

Motivated by these works, we obtain, in this paper, some dark energy isotropic cosmological models of Bianchi type-VI$_0$ with constant and time-dependent EoS parameter. The paper is organized as fallows: We write the metric and field equations in Sect 2. In Sect 3, we obtain the solutions of the field equations. We also discuss the physical behavior of the cosmological models with constant and time-dependent DP and EoS parameters. Finally the concluding remarks are given in Sect 4.

2. Metric and Field Equations

We consider the spatially homogeneous and anisotropic space-time described by Bianchi type VI$_0$ in the form

$$ds^2 = -dt^2 + A^2dx^2 + B^2e^{-2mx}dy^2 + C^2e^{2mx}dz^2 \tag{3}$$

where A, B and C are the scale factors and are functions of cosmic time t, and m is a non-zero constant.
The Einstein's field equations, in natural limits (8πG=c=1) are

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = -T_{\mu\nu} \tag{4}$$

where $R_{\mu\nu}$ is the Ricci tensor, R the scalar curvature and $T_{\mu\nu}$ the energy-momentum tensor of the matter, If the gravitational field is generated by a perfect fluid, the associated energy-momentum tensor has the form

$$T_{\mu\nu} = (\rho + p)v_\mu v_\nu + pg_{\mu\nu} \tag{5}$$

where ρ is the energy density of the cosmic fluid, p is the pressure and v^μ is four velocity vector.

In comoving coordinates $v^\mu = (0, 0, 0, 1)$, the field equations (4), together with (1), yield the following independent equations

$$\frac{\ddot{B}}{B} + \frac{\ddot{C}}{C} + \frac{\dot{B}\dot{C}}{BC} + \frac{m^2}{A^2} = -\omega\rho, \tag{6}$$

$$\frac{\ddot{C}}{C} + \frac{\ddot{A}}{A} + \frac{\dot{C}\dot{A}}{CA} - \frac{m^2}{A^2} = -\omega\rho, \tag{7}$$

$$\frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} + \frac{\dot{A}\dot{B}}{BC} - \frac{m^2}{A^2} = -\omega\rho, \tag{8}$$

$$\frac{\dot{A}\dot{B}}{AB} + \frac{\dot{B}\dot{C}}{BC} + \frac{\dot{C}\dot{A}}{CA} - \frac{m^2}{A^2} = \rho, \tag{9}$$

$$\frac{\dot{B}}{B} - \frac{\dot{C}}{C} = 0, \tag{10}$$

where a dot denotes differentiation with respect to t. The matter, in general, will represent an anisotropic fluid with time-lines as the flow-lines of the fluid. The kinematical parameters the expansion scalar (θ) and shear scalar (σ) of the fluid flow are given by

$$\theta = \frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C}, \tag{11}$$

$$\sigma^2 = \frac{1}{2}\left[\frac{\dot{A}^2}{A^2} + \frac{\dot{B}^2}{B^2} + \frac{\dot{C}^2}{C^2}\right] - \frac{\theta^2}{6}. \tag{12}$$

For the metric (3), the spatial volume V is given by

$$V = ABC. \tag{13}$$

The physical parameters such as directional Hubble's parameters (H_1, H_2, H_3), average Hubble parameter H and the density parameter Ω are defined by

$$H_1 = \frac{\dot{A}}{A}, \qquad H_2 = \frac{\dot{B}}{B}, \qquad H_3 = \frac{\dot{C}}{C},$$

$$H = \frac{1}{3}\left(H_1 + H_2 + H_3\right),$$

$$\Omega = \frac{\rho}{3H^2}. \tag{14}$$

An important observational quantity in cosmology is the deceleration parameter q defined in (2). The sign of q indicates whether the model inflates or not. A positive sign of q corresponds to the standard decelerating model whereas the negative sign of q indicates inflation.

3. Solutions of Field Equations

Solving equation (10), we get

$$B = C \tag{15}$$

where the constant of integration can be absorbed in B or C. Using equation (15), the field equations (6) – (9) reduces to

$$\frac{2\ddot{B}}{B} + \frac{\dot{B}^2}{B^2} + \frac{m^2}{A^2} = -\omega\rho, \tag{16}$$

$$\frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} + \frac{\dot{A}\dot{B}}{AB} - \frac{m^2}{A^2} = -\omega\rho, \tag{17}$$

$$\frac{2\dot{A}\dot{B}}{AB} + \frac{\dot{B}^2}{B^2} - \frac{m^2}{A^2} = -\rho. \tag{18}$$

Thus equations (16) – (18) are three equations in four unknown A, B,ω and ρ. In order to solve the above equations in the closed from, we use a physical condition that the expansion scalar is proportional to shear scalar. According to Thorne [13] observations of velocity redshift relation for extragalactic sources suggest that Hubble expansion of the universe is isotropic about 30% range approximately (Kantowski and Sachs [14], Kristion and Sachs [15] and redshift studies place the limit $\frac{\sigma}{H} \leq 0.30$, Collins [16] discussed the physical significance of this condition for perfect fluid and barotropic equation of state in a more general case. Roy and Prakash [17], Roy and Banerjee [18], Bali and Singh [19] have proposed this condition to find exact solution of cosmological models. Thus, we use the conditions either $B = A^n$ or $A = B^n$, where n is a constant.

3.1 Solutions when $B = A^n$

When $B = A^n$, we obtain from (13) that

$$A = V^{\frac{1}{2n+1}}, \qquad B = V^{\frac{n}{2n+1}} \tag{19}$$

Subtraction of equation (17) from equation (16) gives

$$\frac{\ddot{B}}{B} - \frac{\ddot{A}}{A} + \frac{\dot{B}^2}{B^2} - \frac{\dot{A}\dot{B}}{AB} + \frac{2m^2}{A^2} = 0 \tag{20}$$

Inserting A and B from (19) into (20) we obtain the differential equation for V in the form

$$\ddot{V} = \frac{2m^2(2n+1)}{1-n} V^{\frac{2n-1}{2n+1}} \tag{21}$$

with the solution in the quadrature

$$\int \frac{dV}{V^{\frac{4n}{2n+1}} + C} = \frac{m(2n+1)t}{\sqrt{n(1-n)}} \tag{22}$$

where C is an constant of integration. This equation (22) imposes some restriction on the choice of n, namely $0 < n < 1$. It is difficult to find the solution of the integral (22)in exact form. So, in order to solve the problem completely, we have to choose either C or n in such a manner that (22) becomes integrable.

3.1.1 When $C = 0$

Equation (22) has the solution

$$V = \left(\frac{m}{\sqrt{n(1-n)}}\right)^{2n+1} (t + c_1)^{2n+1} \tag{23}$$

where c_1 is an arbitrary constant. The metric functions A and B are therefore given by

$$A = \frac{m}{\sqrt{n(1-n)}}(t + c_1), \tag{24}$$

$$B = \left(\frac{m}{\sqrt{n(1-n)}}\right)^{n} (t + c_1)^n. \tag{25}$$

The metric of our solution can be written in the form

$$ds^2 = -dt^2 + \frac{m^2}{n(1-n)} T^2 dx^2 + \frac{m^{2n}}{n^n(1-n)^n} T^{2n} (e^{-2mx} dy^2 + e^{2mx} dz^2) \tag{26}$$

where $T = t + c_1$.

For the metric (26), we find that

$$\rho = \frac{n(2n+1)}{T^2}, \tag{27}$$

$$\omega = \frac{1-2n}{1+2n}, \tag{28}$$

$$q = -\frac{2n}{2n+1}, \tag{29}$$

$$\theta = \frac{2n+1}{T}, \tag{30}$$

$$\sigma = \frac{1-n}{\sqrt{3}T}. \tag{31}$$

We observe that for $n = \frac{1}{2}$, $\omega = 0$ which corresponds to dust filled model of the universe. When $0 < n < \frac{1}{2}$, $\omega > 0$ and so we have a perfect fluid filled universe with accelerated expansion. The EoS parameter ω is negative for $\frac{1}{2} < n < 1$, which corresponds to a dark energy model. At $T = 0$ the spatial volume vanishes while all other parameters diverge. Thus the model has a point type singularity at $T = 0$. All physical parameters are decreasing functions of T, and ultimately tend to zero for large T. Since $\frac{\sigma}{\theta}$ is constant, anisotropy is maintained through the passage of time T.

3.1.2 When $C \neq 0$

Equation (22) is integrable for a suitable choice of n which will give a time-dependent DP. The motivation for time-dependent DP is behind the fact that the universe has accelerated expansion at present as observed in recent observation of Type Ia supernova. For a nontrivial solution of equation (22), We choose $n = \frac{1}{2}$, Then equation (22) becomes

$$\int \frac{dV}{\sqrt{V+C}} = 4mt. \tag{32}$$

Equation (32) has the solution given by

$$V = 4m^2t^2 + 2\beta t + \gamma \tag{33}$$

where β and γ are arbitrary constants. From equations (19) and (33), we obtain

$$A = (4m^2t^2 + 2\beta t + \gamma)^{\frac{1}{2}}, \tag{34}$$

$$B = (4m^2t^2 + 2\beta t + \gamma)^{\frac{1}{4}}. \tag{35}$$

The physical parameters such as directional Hubble's parameters, average Hubble parameter,expansion scalar and scale factor a are, respectively given by

$$H_1 = \frac{4m^2t + \beta}{4m^2t^2 + 2\beta t + \gamma}, \tag{36}$$

$$H_2 = H_3 = \frac{4m^2t + \beta}{2(4m^2t^2 + 2\beta t + \gamma)}, \tag{37}$$

$$H = \frac{2(4m^2t + \beta)}{3(4m^2t^2 + 2\beta t + \gamma)}, \tag{38}$$

$$\theta = \frac{2(4m^2t + \beta)}{4m^2t^2 + 2\beta t + \gamma}, \tag{39}$$

$$\sigma = \frac{4m^2t + \beta}{\sqrt{3}(4m^2t^2 + 2\beta t + \gamma)}, \tag{40}$$

$$a = (4m^2t^2 + 2\beta t + \gamma)^{\frac{1}{3}}. \tag{41}$$

The deceleration parameter q has the value given by

$$q = \frac{-8m^2(4m^2t^2 + 2\beta t + \gamma)}{(8m^2t + 2\beta)^2}. \tag{42}$$

In view of equations (33) and (42), q is always negative. The energy density of cosmic fluid, EoS parameter and density parameter Ω are found to be

$$\rho = \frac{(8m^2t + 2\beta)^2 + (\beta^2 - 4m^2\gamma)}{4(4m^2t^2 + 2\beta t + \gamma)^2}, \tag{43}$$

$$\omega = \frac{-5(4m^2\gamma - \beta^2)}{(8m^2t + 2\beta)^2 - (4m^2\gamma - \beta^2)}, \tag{44}$$

$$\Omega = \frac{3}{16}\frac{(8m^2t + 2\beta)^2 + (\beta^2 - 4m^2\gamma)}{(4m^2t + \beta)^2}. \tag{45}$$

The EoS parameter ω is negative if $\gamma > \frac{\beta^2}{4m^2}$. If this condition holds, the model has no finite singularity. In this case, we obtain a non-singular dark energy model of the current accelerated expansion universe. If $\gamma < \frac{\beta^2}{4m^2}$, $\omega > 0$ which leads to an accelerating perfect fluid model of the universe. The physical and kinematical behaviors of this models are same as (26).

4. Conclusion

In this paper, we have constructed Bianchi type-VI$_0$ cosmological models with dark energy under the assumption that the shear scalar is proportional to the expansion scalar. We have followed the procedure of Saha and Yadav (2012) for solving Einstein's field equations. Under some specific choice of problem parameters the present consideration yields constant and variable DP and EoS parameters. In the case of constant DP and EoS parameters we have obtained a dark energy cosmological model with singularity at the initial time, whereas the model with variable DP and EoS parameters has no finite singularity. In both types of models anisotropy is maintained throughout the passage of time. From the theoretical perspective the present models can be viable models to explain the late time acceleration of the universe.

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