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Towards a Geometrodynamic Entropic Approach to Quantum Entanglement and the Perspectives on Quantum Computing

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Abstract: On the basis of the Fisher Bohm Entropy, we outline here a geometric approach to quantum information. In particular, we review the spin-spin correlation and berry Phase, and study the processes of coherence and decoherence in terms of the number of thermal microstates of the qubits system

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1. Introduction

One of the essential features of the geometry of quantum theory lies in the fact that subatomic particles show space-like correlations and can be in entangled states. Although it has been experimentally verified, the phenomenon of entanglement – which apparently does not have a classical counterpart – continues to be mysterious and paradoxical within the usual quantum treatment and remains a focal point for discussing the foundations of quantum phenomena [1]. In particular, it seems is tightly related with the inherent non-locality of the interactions between subatomic particles and with the introduction of Bell's inequality, as well as with the matter of the existence of hidden variables [2, 3].

Today we assist at a growing interest and activity as regards the entanglement because of the impetus derived from the recent developments in quantum information and quantum technology. Recent research demonstrate the crucial role of the quantum entanglement as a vital resource in quantum information through quantum teleportation [4],

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quantum cryptographic key distribution [5] and quantum computation [6, 7] as well as its potential for enhanced quantum sensing through the engineering of highly entangled quantum states, beating the usual quantum limit [8].

As regards the geometry subtended by entanglement, in the paper "Subquantum information and computation", A. Valentini argues that "*immense physical resources are hidden from us by quantum noise, and that we will be unable to access those resources only for as long as we are trapped in the 'quantum heat death' – a state in which all systems are subject to the noise associated with the Born probability distribution* $\rho = |\psi|^2$ "[9]. As clearly showed by Valentini, hidden-variables theories offer a radically different perspective on quantum information theory. In such theories, a huge amount of 'subquantum information' is hidden from us simply because we happen to live in a time and place where the hidden variables have a certain 'equilibrium' distribution.

De Broglie's pilot-wave approach, according to which subatomic particles are point-like objects with continuous trajectories "guided" or choreographed by the wave function [10, 11, 12] suggests interesting new perspectives about quantum information: one deals here with a security of quantum cryptography which depends on our being trapped in quantum equilibrium; and, non-equilibrium would unleash computational resources far more powerful than those of quantum computers. Although some might prefer to regard this work as showing how the principles of quantum information theory depend on a particular axiom of quantum theory – the Born rule $\rho = |\psi|^2$, if one takes hidden-variables theories seriously as physical theories of nature, one can hardly escape the conclusion that we just happen to be confined to a particular state in which our powers are limited by an all-pervading statistical noise. On the light of the geometry of the quantum world determined by de Broglie's approach, it then seems important to search for violations $\rho \neq |\psi|^2$ of the Born rule [13-16].

De Broglie's pilot wave theory was developed in detail since the 50's of last century by David Bohm who was the first to realize that the concept of pilot wave represents a foundation of quantum mechanics [17, 18]. Bohm's interpretation of quantum mechanics throws new light into the analysis of quantum entanglement by the introduction, along with hidden variables, of a non-local quantum potential. While in the Copenhagen interpretation of quantum mechanics non-locality emerges as an unexpected host which lies behind the purely probabilistic interpretation of the wave function and the mechanism of "casuality" associated with it, Bohm's approach is explicitly non-local in virtue of the space-like, active, contextual information of the quantum potential.

In Bohm's interpretation, the quantum potential emerges as the ultimate visiting card which allows us to obtain predictions in accordance with standard quantum mechanics and, at the same time, a description of the behaviour of subatomic particles in agreement with the principle of causality and the motion dogma. Since the introduction of Bohm's interpretation of quantum mechanics, it has been in detail explored how this approach leads to outcome predictions of various experiments identical to the results of standard quantum mechanics, and at the same time is able to analyse individual processes in a way which goes beyond the standard interpretation, providing a description of the motion of particles in terms of particular coordinates – commonly termed as the Bohm hidden variables. Bohm's approach was also extended from original single spinless particles to many-body fermionic or bosonic systems [19, 20, 21] and also to the quantum field theory, including creation and annihilation of particles [22, 23]. Moreover, recently it was shown how the Born-rule probability densities of non-relativistic quantum mechanics emerge naturally from the particle dynamics of de Broglie-Bohm theory [24]. The interpretation of several typically quantum experiments was developed, for example, for the double slit experiments or tunnelling of particles through barriers [19], the Aharonov-Bohm effect [25], and Stern-Gerlach experiments [26]. The Bohmian theory in terms of well-defined individual particle trajectories with continuously variable spin vectors was successfully applied also to the problem of Einstein-Podolsky-Rosen spin correlations [27]. Among the simplest quantum objects are two state systems and the formal Bohm approach is consistently extended to spin $1/2$ systems with non-relativistic formalism based on the Pauli equation [28], causal rigid rotor theory [29], via spinor wave functions [20], the Bohm-Dirac model for entangled electrons [30], by Clifford algebra approach to Schrödinger [31] or relativistic Dirac particles [32].

In this paper we want to introduce new perspectives as regards quantum entanglement and to derive consequences in quantum computing by starting from a geometrodynamic entropic approach to quantum information – in the picture of Bohm's quantum potential under a minimum condition of Fisher metric – proposed recently in [33]. The paper is structured in the following manner. In chapter 2 we review the geometrodynamic entropic approach to quantum information. In chapter 3 we study the spin-spin correlations of entangled qubit. In chapter 4, we provide a geometrodynamic entropic model for the Berry phase. Finally, in chapter 5 we conclude with some considerations about the perspectives of the geometrodynamic entropic approach in quantum computing.

2. The Geometrodynamic Entropic Approach to Quantum Information

In the recent paper [33] the authors introduced a new notion of geometric quantum information in a quantum entropy space in which the quantum potential expresses how the quantum effects deform the configuration space of processes in relation to the number of the microstates of the system under consideration. In this geometrodynamic approach, the quantum potential appears under the constraint of a minimum condition of Fisher information as a non-Euclidean deformation of the information space associated with the quantum entropy given by the vector of the superposition of different Boltzmann entropies

$$\begin{cases} S_1 = k \log W_1(\theta_1, \theta_2, ..., \theta_p) \\ S_2 = k \log W_2(\theta_1, \theta_2, ..., \theta_p) \\ ... \\ S_n = k \log W_n(\theta_1, \theta_2, ..., \theta_p) \end{cases} \tag{1}$$

where W are the number of the microstates of the system for the same parameters θ as temperatures, pressures, etc. . . .. In this picture, quantum effects are equivalent to a geometry which is described by the following equation

$$\frac{\partial}{\partial x^k} + \frac{\partial^2 S_j}{\partial x^k \partial x^p} \frac{\partial x^i}{\partial S_j} = \frac{\partial}{\partial x^k} + \frac{\partial \log W_j}{\partial x_h} = \frac{\partial}{\partial x^k} + B_{j,h} \tag{2}$$

where

$$B_{j,h} = \frac{\partial S_j}{\partial x_h} = \frac{\partial \log W_j}{\partial x_h} \tag{3}$$

is a Weyl-like gauge potential [21, 22]. The change of the geometry is associated with a deformation of the moments stated by the action

$$A = \int \rho \left[\frac{\partial A}{\partial t} + \frac{1}{2m} p_i p_j + V + \frac{1}{2m} \left(\frac{\partial \log W_k}{\partial x_i} \frac{\partial \log W_k}{\partial x_j} \right) \right] dt d^n x \tag{4}$$

The quantum action assumes the minimum value when $\delta A = 0$ namely

$$\delta \int \rho \left[\frac{\partial A}{\partial t} + \frac{1}{2m} p_i p_j + V \right] dt d^n x + \delta \int \frac{\rho}{2m} \frac{\partial \log W_k}{\partial x_i} \frac{\partial \log W_k}{\partial x_j} dt d^n x = 0 \tag{5}$$

and thus

$$\frac{\partial A}{\partial t} + \frac{1}{2m} p_i p_j + V + \frac{1}{2m} \left(\frac{1}{W_k^2} \frac{\partial W_k}{\partial x_i} \frac{\partial W_k}{\partial x_j} - \frac{2}{W_k} \frac{\partial^2 W_k}{\partial x_i \partial x_j} \right) = \frac{\partial S_k}{\partial t} + \frac{1}{2m} p_i p_j + V + Q \tag{6}$$

where Q is the Bohm quantum potential that is a consequence for the extreme condition of Fisher information. On the basis of equation (6), one can interpret Bohm's quantum potential as an information channel determined by the functions W_k defining the number of microstates of the physical system under consideration, which depend on the parameters θ of the distribution probability (and thus, for example, on the space-temporal distribution of an ensemble of particles, namely the density of particles in the element of volume d^3x around a point $\vec{x}$ at time t) and which correspond to the vector of the superpose Boltzmann entropies (1). In other words, in this approach, the distribution probability of the wave function determines the functions W_k defining the number of microstates of the physical system under consideration, a quantum entropy emerges from these functions W_k given by equations (1), and these functions W_k, and therefore the quantum entropy itself (namely the vector of the superpose entropies), can be considered as the fundamental physical entities that determine the action of the quantum potential (in the extreme condition of the Fisher information) on the basis of equation[2]

$$Q = \frac{1}{2m}\left(\frac{1}{W^2}\frac{\partial W}{\partial x_i}\frac{\partial W}{\partial x_j} - \frac{2}{W}\frac{\partial^2 W}{\partial x_i \partial x_j}\right). \tag{7}$$

In this way, under the constraint of the minimum condition of Fisher information, the Boltzmann entropies defined by equations (1) emerge as "informational lines" of the quantum potential. In other words, each of the entropies appearing in the superposition vector (1) can be considered as a specific information channel of the quantum potential. on the basis of the picture based on equations (1)-(7), the difference between classical and quantum information is similar to the difference between Euclidean and non-Euclidean geometry in the parameter space determined by the quantum entropy. Superposition quantum states are described by the deformation of the geometry associated with the quantum entropy [1, 2].

Moreover, on the basis of equations (2) and (6), one can say that the change of the geometry in the presence of quantum effects, which is expressed by a Weyl-like gauge potential is determined by the functions W and thus by the quantum entropy. It becomes so permissible the following reading of the mathematical formalism in non relativitstic bohmian quantum mechanics: the distribution probability of the wave function determines the functions W defining the number of microstates of the physical system under consideration, a quantum entropy emerges from these functions W given by equations (1), and these functions W (and thus also the quantum entropy given by equations (1)) determine a change of the geometry expressed by a Weyl-like gauge potential and characterized by a deformation of the moments given by equation (4). The quantum potential, in extreme condition of Fisher information, can be therefore considered as an information channel describing the change of the geometry of the physical space in the presence of quantum effects.

Now, by introducing the definition (7) of the quantum potential as an informational entity about the geometry, inside the quantum Hamilton-Jacobi equation, we obtain

$$\frac{|\nabla S|^2}{2m} + V + \frac{1}{2m}\left(\frac{1}{W^2}\frac{\partial W}{\partial x_i}\frac{\partial W}{\partial x_j} - \frac{2}{W}\frac{\partial^2 W}{\partial x_i \partial x_j}\right) = -\frac{\partial S}{\partial t} \tag{8}$$

which provides a new way to read the energy conservation law in quantum mechanics. In equation (8) two quantum corrector terms appear in the energy of the system, which are owed to the functions W linked with the quantum entropy, and which thus describe the deformation, the change of the geometry in the presence of quantum effects. On the basis of equation (8), we can say that, in the approach here suggested, the distribution probability of the wave function determines the functions W defining the number of microstates of the physical system under consideration, a quantum entropy emerges from these functions W given by equations (1), these functions W (and thus also the quantum

[2] In the next equations of chapter 2, for simplicity we are going to denote the generic function W_k (defined by equations 10) with W.

entropy given by equations (1)) determine a deformation, a change of the geometry of the physical space, and this change of the geometry produces two quantum corrector terms in the energy of the system. These two quantum corrector terms can thus be interpreted as a sort of modification in the background space, as a sort of degree of chaos of the background space determined by the ensemble of particles associated with the wave function under consideration. According to the approach here suggested, it is just the functions W and thus the quantum entropy the fundamental entities which, by corresponding with the degree of chaos of the background space determined by the ensemble of particles associated with the wave function under consideration, represent what are the geometric properties of space from which the quantum force, and thus the behaviour of quantum particles, derive. In non-relativistic bohmian mechanics, the geometrodynamic nature of the quantum potential (namely the fact that it has a geometric nature, contains a global information on the environment in which the experiment is performed, and at the same time it is a dynamical entity, namely its information about the process and the environment is active) is just determined by the deformation of the geometry determined by the functions W (and thus by the quantum entropy) on the basis of equations (7) and (8). The functions W emerge just as informational lines of the quantum potential describing the change of the geometry of the physical space in the presence of quantum effects. It is also interesting to observe that, in this approach, the inverse square root of the quantity

$$L_{quantum} = \frac{1}{\sqrt{\frac{1}{\hbar^2}\left(\frac{2}{W}\frac{\partial^2 W}{\partial x_i \partial x_j} - \frac{1}{W^2}\frac{\partial W}{\partial x_i}\frac{\partial W}{\partial x_j}\right)}} \tag{9}$$

defines a typical quantum-entropic length that can be used to evaluate the strength of quantum effects and, therefore, the modification of the geometry with respect to the Euclidean geometry characteristic of classical physics. Once the quantum-entropic length becomes non-negligible the system goes into a quantum regime. In this picture, Heisenberg's uncertainty principle derives from the fact that we are unable to perform a classical measurement to distances smaller than the quantum-entropic length. In other words, the size of a measurement has to be bigger than the quantum-entropic length

$$\Delta L \geq L_{quantum} = \frac{1}{\sqrt{\frac{1}{\hbar^2}\left(\frac{2}{W}\frac{\partial^2 W}{\partial x_i \partial x_j} - \frac{1}{W^2}\frac{\partial W}{\partial x_i}\frac{\partial W}{\partial x_j}\right)}}. \tag{10}$$

The quantum regime is entered when the quantum-entropic length must be taken under consideration.

In synthesis, inside the entropic approach to Bohm's quantum potential, we can say that by means of a Fisher geometry it is possible to characterize the deformations of physical space in the presence of quantum effects in the space of parameters and to express Bohm's quantum potential as the information channel indicating the modification of the geometry of physical space determined by the quantum entropy. Novello, Salim and Falciano [24] have recently proposed a geometrical approach in which the presence of quantum effects is linked with the Weyl length $L_W = \frac{1}{\sqrt{R}}$, and thus with the curvature scalar. In analogy with Novello's, Salim's and Falciano's approach that implies that the quantum effects are the manifestations of the modification of the structure of the three-dimensional physical space from Euclidean to a non-Euclidean Weyl integrable space, inside the geometrodynamic entropic approach proposed by the authors, quantum effects are linked with a change in the geometrical properties of the background, which is determined by the quantum entropy and therefore their strength is described by the quantum entropy.

On the basis of the entropic approach to quantum potential developed in this chapter, it is possible now to provide a new reading to the active information of quantum potential and thus to throw new fundamental light into the interpretation of quantum information. Inside this approach, since in the extreme condition of Fisher information the quantum potential is an information channel which describes the deformation of the geometry of the physical space in the presence of quantum effects and is determined by the quantum entropy, the real visiting card, the real ultimate grid of quantum information can be considered the deformation of the background space determined by the quantum entropy. The vector of the superposition of different Boltzmann entropies (10) emerges thus as the most fundamental source of quantum information.

3. Spin-spin Correlations in Entangled Qubit Pairs in the Geometrodynamic Entropic Approach

A two-state quantum system can be regarded as a qubit. When one qubit interplays with the other qubit, the interplay can also be modelled by the exchange-tipe interactions of spins. In this chapter our aim is to provide a geometrodynamic approach of a qubit pair of spin 1/2 particles in a pure state

$$|\psi\rangle = \cos\frac{\vartheta}{2}\left|\uparrow\downarrow\right\rangle + e^{i\varphi}\sin\frac{\vartheta}{2}\left|\downarrow\uparrow\right\rangle \tag{11}$$

where $|\uparrow\downarrow\rangle$ corresponds to the state of the system when the first particle (qubit) is in the "up" state, i.e., in the direction of the z-axis, and the second qubit is in the "down" state, while $|\downarrow\uparrow\rangle$ corresponds to the state of the system when the first qubit is the down state and the second qubit is in the up state. Qubits are not restricted to real electron spins, but may be realized by any two state quantum system such as for example, entangled photon [, flux qubit in a superconducting ring [34], charge pseudo-spin of electron pairs in a double quantum dot [35], flying qubits in quantum point contacts [36] or qubits in a composite system [37].

In the papers [19, 29] Holland showed that quantum entanglement can be described in a bohmian framework by starting from the mapping between a quantum spherically symmetric rigid rotor and a classical spinning top in the presence of a quantum potential. Defining a differential operator $\hat{\vec{M}}$ representing the angular momentum, whose components are the infinitesimal generators of the rotation group $SO\,(3)$, the Hamiltonian of a quantum spherically symmetric rigid rotor is given by

$$\hat{H} = \frac{\hat{\vec{M}}^2}{2I} \tag{12}$$

where I is the moment of inertia. Eigenstates of the three mutually commuting operators $\hat{\vec{M}}^2$, M_z and $\vec{e}\cdot\hat{\vec{M}}$ are functions of Euler angles $\xi = (\alpha, \beta, \gamma)$, specifying the orientation of a rigid body with the principal axis defined by a normalized vector $\vec{e}$. The wave function is expressed as $\psi = Re^{iS}$, where $R(\xi)$ and $S(\xi)$ are real functions. Bohmian space angular momentum is then given by a real three dimensional vector

$$\vec{M} = i\hat{\vec{M}}S. \tag{13}$$

Relation (13) is an analogue of a more familiar de Broglie's guidance equation for the velocity of a point-like particle with mass m treated in the Bohmian approach, $m\vec{v} = \nabla S$ [17].

The dynamics is determined from the Hamilton-Jacobi-type equations for the classical Hamiltonian with an additional quantum potential Q,

$$\hat{H} = \frac{\hat{\vec{M}}^2}{2I} + Q, \tag{14}$$

$$Q = \frac{\hat{\vec{M}}^2 R}{2IR} \tag{15}$$

where the quantum potential generates a quantum torque

$$\vec{T} = -i\hat{\vec{M}}Q \tag{16}$$

which rotates the angular momentum vector via the equation of motion

$$\frac{d\vec{M}}{dt} = \vec{T} \tag{17}$$

along the trajectory $\xi(t)$. This is a counterpart of the Newton equation for the case of a free particle in the Bohm formulation given by

$$m\frac{d\vec{v}}{dt} = \nabla\left(\frac{\nabla^2 R}{2mR}\right). \tag{18}$$

The equation of the angular momentum motion simplifies to a set of first order non-linear differential equations for the trajectories in the configuration space

$$I\dot{\alpha} = \frac{\partial S}{\partial \alpha} \tag{19}$$

$$I\dot{\beta} = \left(\frac{\partial S}{\partial \beta} - \cos\alpha\frac{\partial S}{\partial \gamma}\right)/\sin^2\alpha \tag{20}$$

$$I\dot{\gamma} = \left(\frac{\partial S}{\partial \gamma} - \cos\alpha \frac{\partial S}{\partial \beta}\right) / \sin^2 \alpha \tag{21}$$

where the solutions $\xi(t)$ represent orbits in the configuration space, uniquely determined by the initial positions $\xi(0) = (\alpha_0, \beta_0, \gamma_0)$, and the angular momentum emerges as $\vec{M}[\xi(t)]$.

Now, in the geometrodynamic entropic approach one assumes that the quantum effects of a symmetric rigid rotor can be described by a change of the geometrical properties of the configuration space associated with the quantum entropy (1) which is associated with a deformation of the angular momenta stated by the action

$$A = \int \rho \left[\frac{\partial A}{\partial t} + \frac{1}{2I} M_i M_j + V + \frac{1}{2I}\left(\frac{\partial \log W_k}{\partial x_i}\frac{\partial \log W_k}{\partial x_j}\right)\right] dt d^n x. \tag{22}$$

The quantum action assumes the minimum value when

$$\delta \int \rho \left[\frac{\partial A}{\partial t} + \frac{1}{2I} M_i M_j + V\right] dt d^n x + \delta \int \frac{\rho}{2I} \frac{\partial \log W_k}{\partial x_i}\frac{\partial \log W_k}{\partial x_j} dt d^n x = 0 \tag{23}$$

namely

$$\frac{\partial A}{\partial t} + \frac{1}{2I} M_i M_j + V + \frac{1}{2I}\left(\frac{1}{W_k^2}\frac{\partial W_k}{\partial x_i}\frac{\partial W_k}{\partial x_j} - \frac{2}{W_k}\frac{\partial^2 W_k}{\partial x_i \partial x_j}\right) = \frac{\partial S_k}{\partial t} + \frac{1}{2I} M_i M_j + V + Q. \tag{24}$$

where Q is the quantum potential (15) describing the dynamics of a spherically symmetric rigid rotor, that emerges as a consequence for the extreme condition of Fisher information. The quantum potential of a spherically symmetric rigid rotor becomes therefore

$$Q = \frac{\hat{\vec{M}}^2 R}{2IR} = \frac{1}{2I}\left(\frac{1}{W_k^2}\frac{\partial W_k}{\partial x_i}\frac{\partial W_k}{\partial x_j} - \frac{2}{W_k}\frac{\partial^2 W_k}{\partial x_i \partial x_j}\right) \tag{25}$$

and thus one obtains

$$\hat{\vec{M}}^2 R = \left(\frac{1}{W_k^2}\frac{\partial W_k}{\partial x_i}\frac{\partial W_k}{\partial x_j} - \frac{2}{W_k}\frac{\partial^2 W_k}{\partial x_i \partial x_j}\right) R. \tag{26}$$

According to the formalism based on equations (22)-(26), one can say that the quantum effects associated with the angular momentum are determined by the functions W_k defining the number of microstates of the rigid rotor, which depend on the parameters θ of the distribution probability (and thus, for example, on the space-temporal distribution of an ensemble of particles, namely the density of particles in the element $d^3\xi$ along the trajectory $\xi(t)$) and which correspond to the vector of the superpose Boltzmann entropies (1). In other words, the functions W_k and therefore the Boltzmann entropies can be considered as informational lines of the quantum potential of the rigid rotor. Moreover, these functions W_k and therefore the Boltzmann entropies determine a deformation of the background of the processes in the sense that they generate a quantum torque

$$\vec{T} = -\frac{i}{2I}\left(\frac{1}{W_k^2}\frac{\partial W_k}{\partial x_i}\frac{\partial W_k}{\partial x_j} - \frac{2}{W_k}\frac{\partial^2 W_k}{\partial x_i \partial x_j}\right)\hat{\vec{M}} \tag{27}$$

which rotates the angular momentum vector via the equation of motion

$$\frac{d\vec{M}}{dt} = -\frac{i}{2I}\left(\frac{1}{W_k^2}\frac{\partial W_k}{\partial x_i}\frac{\partial W_k}{\partial x_j} - \frac{2}{W_k}\frac{\partial^2 W_k}{\partial x_i \partial x_j}\right)\hat{\vec{M}}. \tag{28}$$

along the trajectory $\xi(t)$. It is also interesting to observe that, in this approach, the modification of the geometry associated with the functions W_k and thus with the quantum entropy as regards a spherically symmetric rigid rotor can be characterized introducing the quantum-entropic length given by relation

$$L_{quantum} = \frac{1}{\sqrt{\frac{1}{2I\hbar^2}\left(\frac{2}{W}\frac{\partial^2 W}{\partial x_i \partial x_j} - \frac{1}{W^2}\frac{\partial W}{\partial x_i}\frac{\partial W}{\partial x_j}\right)}}. \tag{29}$$

Once the quantum-entropic length becomes non-negligible the rigid rotor goes into a quantum regime.

Let us analyse now the dynamics and the quantum effects of a general two qubits state, with vanishing total angular momentum projections, given by equation (11). In a bohmian framework the guiding wave function

$$\psi(\xi) = \cos\frac{\vartheta}{2}u_\uparrow(\xi_1)\,u_\downarrow(\xi_2) + e^{i\varphi}\sin\frac{\vartheta}{2}u_\downarrow(\xi_1)\,u_\uparrow(\xi_2) \tag{30}$$

is given in a six-dimensional space spanned by $\xi = \{\xi_1, \xi_2\}$, where ξ_1, ξ_2 are the coordinates of the first and the second rotor, respectively. The corresponding Hamiltonian is given by relation

$$H = \frac{\vec{M}_1^2 + \vec{M}_2^2}{2I} + Q \tag{31}$$

where

$$Q = \frac{\left(\hat{M}_1^2 + \hat{M}_2^2\right)R}{2IR} \tag{32}$$

is the quantum potential. It is interesting to observe that, even for two non-interacting, but entangled qubits, the quantum potential (32) represents an instant interaction between the rotors as a fingerprint of the quantum nature of the problem. The solutions for each of the angular momentum vectors $\vec{M}_1$ and $\vec{M}_2$ are functions of six common coordinates forming the trajectory $\xi(t)$ determined by six initial values $\xi(0)$. In this case of two entangled qubits, the extreme condition of Fisher information is

$$\begin{aligned}&\tfrac{\partial A}{\partial t} + \tfrac{1}{2I}M_{i_1}M_{j_1} + \tfrac{1}{2I}M_{i_2}M_{j_2} + V + \tfrac{1}{2I}\left(\tfrac{1}{W_{k_1}^2}\tfrac{\partial W_{k_1}}{\partial x_{i_1}}\tfrac{\partial W_{k_1}}{\partial x_{j_1}} - \tfrac{2}{W_{k_1}}\tfrac{\partial^2 W_{k_1}}{\partial x_{i_1}\partial x_{j_1}}\right) + \\ &\tfrac{1}{2I}\left(\tfrac{1}{W_{k_2}^2}\tfrac{\partial W_{k_2}}{\partial x_{i_2}}\tfrac{\partial W_{k_2}}{\partial x_{j_2}} - \tfrac{2}{W_{k_2}}\tfrac{\partial^2 W_{k_2}}{\partial x_{i_2}\partial x_{j_2}}\right) = \tfrac{\partial S_k}{\partial t} + \tfrac{1}{2I}M_iM_j + V + Q\end{aligned} \tag{33}$$

and thus the quantum potential may be expressed as

$$Q = \frac{1}{2I}\left(\frac{1}{W_{k_1}^2}\frac{\partial W_{k_1}}{\partial x_{i_1}}\frac{\partial W_{k_1}}{\partial x_{j_1}} - \frac{2}{W_{k_1}}\frac{\partial^2 W_{k_1}}{\partial x_{i_1}\partial x_{j_1}} + \frac{1}{W_{k_2}^2}\frac{\partial W_{k_2}}{\partial x_{i_2}}\frac{\partial W_{k_2}}{\partial x_{j_2}} - \frac{2}{W_{k_2}}\frac{\partial^2 W_{k_2}}{\partial x_{i_2}\partial x_{j_2}}\right) \tag{34}$$

and thus one has

$$\left(\hat{M}_1^2+\hat{M}_2^2\right)R=\left(\frac{1}{W_{k_1}^2}\frac{\partial W_{k_1}}{\partial x_{i_1}}\frac{\partial W_{k_1}}{\partial x_{j_1}}-\frac{2}{W_{k_1}}\frac{\partial^2 W_{k_1}}{\partial x_{i_1}\partial x_{j_1}}+\frac{1}{W_{k_2}^2}\frac{\partial W_{k_2}}{\partial x_{i_2}}\frac{\partial W_{k_2}}{\partial x_{j_2}}-\frac{2}{W_{k_2}}\frac{\partial^2 W_{k_2}}{\partial x_{i_2}\partial x_{j_2}}\right)R. \tag{35}$$

Moreover, the modification of the geometry associated with the entangled qubit pair can be described introducing the quantum-entropic length given by relation

$$L_{quantum}=\frac{1}{\sqrt{\frac{1}{2I\hbar^2}\left(\frac{2}{W_1}\frac{\partial^2 W_1}{\partial x_{i_1}\partial x_{j_1}}-\frac{1}{W_1^2}\frac{\partial W_1}{\partial x_{i_1}}\frac{\partial W_1}{\partial x_{j_1}}+\frac{2}{W_2}\frac{\partial^2 W_2}{\partial x_{i_2}\partial x_{j_2}}-\frac{1}{W_2^2}\frac{\partial W_2}{\partial x_{i_2}}\frac{\partial W_2}{\partial x_{j_2}}\right)}}. \tag{36}$$

As regards a quibit pair in the state parametrized by equation (11), the total angular momentum projection $M_{1z}+M_{2z}$ is zero while the angular momenta due to the action of the non-local quantum potential (34) and the corresponding quantum torques

$$\vec{T}_1=-\frac{i}{2I}\left(\frac{1}{W_{k_1}^2}\frac{\partial W_{k_1}}{\partial x_i}\frac{\partial W_{k_1}}{\partial x_j}-\frac{2}{W_{k_1}}\frac{\partial^2 W_{k_1}}{\partial x_i\partial x_j}+\frac{1}{W_{k_2}^2}\frac{\partial W_{k_2}}{\partial x_i}\frac{\partial W_{k_2}}{\partial x_j}-\frac{2}{W_{k_2}}\frac{\partial^2 W_{k_2}}{\partial x_i\partial x_j}\right)\hat{\vec{M}}_1 \tag{37}$$

and

$$\vec{T}_2=-\frac{i}{2I}\left(\frac{1}{W_{k_1}^2}\frac{\partial W_{k_1}}{\partial x_i}\frac{\partial W_{k_1}}{\partial x_j}-\frac{2}{W_{k_1}}\frac{\partial^2 W_{k_1}}{\partial x_i\partial x_j}+\frac{1}{W_{k_2}^2}\frac{\partial W_{k_2}}{\partial x_i}\frac{\partial W_{k_2}}{\partial x_j}-\frac{2}{W_{k_2}}\frac{\partial^2 W_{k_2}}{\partial x_i\partial x_j}\right)\hat{\vec{M}}_2 \tag{38}$$

exhibit a complex precessional motion.

In the papers "Geometrical view of quantum entanglement" [39] and "Spin-spin correlations of entangled qubit pairs in the Bohm interpretation of quantum mechanics" [40], Ramsak showed how quantum entanglement of a pair of qubits may be visualized in geometrical terms: by analysing the dynamics of the qubits in the framework of the de Broglie-Bohm interpretation of quantum mechanics, he found that the angular momenta of two qubits can be viewed geometrically in the bohmian space of hidden variables and characterized by their relative angles. For perfectly entangled pairs, the qubits exhibit a unison precession making a constant angle between their angular momenta. In particular, in the paper [39] Ramsak computed trajectories $\vec{M}_{1,2}\left[\xi\left(t\right)\right]$ covering the full configuration space with $\approx 10^6$ initial values $\xi\left(0\right)$ per $|\psi\rangle$, i.e. for a particular choice of ϑ and φ. Although these trajectories exhibit extremely rich variety, the following common properties can be outlined:

(1) The quasi-periodic motion appears chaotic and, except in special cases, the projections of the total momentum M onto the xy-plane winds around the origin an infinite number of times in a spirographic manner, forming a dense annulus limited by fixed outer and inner radii;

(2) The curve corresponding to relative momentum $\vec{M}_2-\vec{M}_1$ is closed and periodic.

These results suggest that the entanglement properties of a qubit pair in the state (11) are linked with the dynamics of the azhimutal angles $\phi_1\left[\xi\left(t\right)\right]$ and $\phi_2\left[\xi\left(t\right)\right]$ of the angular momenta. In the Bohmian interpretation, the angular momentum vectors of the two particles (qubits) precess in a well-defined way with some initial probability distribution. In particular, on the basis of Ramsak's results, the probability distribution

$$\frac{dP(\phi)}{d\phi} = \int \delta[\phi - \phi(\xi)] R^2(\xi) d\xi \tag{39}$$

of the ensemble average difference of azimuthal angles $\phi[\xi(t)] = \phi_2 - \phi_1$ is constant for unentangled qubits and becomes progressively peaked at φ for increasing entanglement, culminating in precession of angular momenta at equal relative angle $\phi[\xi(t)] = \varphi$ for all ξ consistent with perfect entanglement. The shape of the distribution is independent of φ.

In the approach here suggested the crucial point is the following. The behaviour of the probability distribution (39) is determined by the quantum torques (37) and (38) and thus by the functions W_{k_1} and W_{k_2}, defining the number of microstates of the particle 1 and particle 2 of the system respectively (and representing the informational lines of the quantum potential (34)). The entanglement properties of a pair of particles in the state (11) are therefore determined by the deformation of the geometrical properties of the background produced by the quantum torques (37) and (38). The geometrical properties of the background of the two particles into consideration can also be characterized by defining the following quantum entropy for a two-qubit system

$$\begin{cases} S_1 = k\log(W_{1_1}W_{1_2}) \\ S_2 = k\log(W_{2_1}W_{2_2}) \\ \cdots\cdot \\ \cdots\cdot \\ S_n = k\log(W_{n_1}W_{n_2}) \end{cases} \tag{40}$$

The peaking of the probability distribution (39) for increasing entanglement between the two qubits is associated with the precessional motion of the quantum torques (37) and (38) and thus with the informational lines of the quantum potential of the system of the two qubits. In the approach based on equations (33)-(38), it becomes the following re-reading of Ramsak's results listed above, and in particular of the behaviour of the probability distribution (39): as regards a system of two entangled qubits the extreme condition of the Fisher metric determines the quantum potential (34) which corresponds with the deformation of the geometry described by the quantum-entropic length (36) produced by the quantum entropy, and the quantum information associated with the quantum potential (34) and thus the quantum-entropic length (36) generate the quantum torques (37) and (38) which, by exhibiting a precessional motion, imply the precession of angular momenta at equal relative angle $\phi[\xi(t)] = \varphi$ for all ξ consistent with perfect entanglement regarding the probability distribution (39). The dynamics of the azimuthal angles $\phi_1[\xi(t)]$ and $\phi_2[\xi(t)]$ of angular momenta – and, consequently, the chaotic features of the trajectories in which the projections of the total momentum M onto the xy-plane winds around the origin an infinite number of times in a spirographic manner and the closed and periodic curve corresponding to the relative momentum $\vec{M}_2-\vec{M}_1$ – are therefore determined by the quantum torques (37) and (38), and thus by the quantum potential (34), namely by the deformation of the geometrical properties of the background described by the quantum entropy (40).

Moreover, by classifying the ensemble representatives of the system of two qubits on the basis of the relative direction of angular momenta precession of the two qubits, Ramsak found that in the low entanglement regime the xy-plane projection of momenta $\vec{M}_1$ and $\vec{M}_2$ precess mainly in opposite directions, and that, in general, momentum pairs move part time in the same and part time in the opposite direction. Ramsak defined representatives that always precess in the same direction as "concurrent" movers whereas those which always precess in the opposite direction as "anticoncurrent" and measured the concurrency of the trajectories $\xi(t)$ by introducing the quantity

$$C_\xi = \frac{1}{\tau}\int_0^\tau sign\frac{d\phi_1[\xi(t)]}{dt}\frac{d\phi_2[\xi(t)]}{dt}dt. \tag{41}$$

At each moment the angular momenta for a given trajectory $\xi(t)$ precess either in the same or in the opposite direction. The concurrency of a trajectory $\xi(t)$ can be visualized as a measure of the share of the time that both angular momenta move in the same direction. For example, one has $C_\xi = \pm 1$ for perfectly concurrent and anti-concurrent movers, respectively, and $C_\xi > 0$ for trajectories where angular momenta move concurrently more than half of the time for some members of the ensemble. In the low-entanglement regime anticoncurrent movers dominate whereas the distribution of concurrent movers progressively dominates as entanglement increases. The concurrency (41) is characterized by the probability distribution

$$\frac{dP(C_\xi)}{dC_\xi} = P_+\delta(C_\xi - 1) + P_-\delta(C_\xi + 1) + \rho(C_\xi) \tag{42}$$

where $P_\pm$ is the probability that the concurrency is exactly ± 1, respectively, and $\rho(C_\xi)$ is a continuous function for which motion is sometimes concurrent and sometimes anti-concurrent as t changes. In the context of the geometrodynamic entropic approach based on equations (33)-(38), the concurrency (41) of the trajectories of the angular momentum vectors and their distribution probability (42) can be seen as a consequence of the behaviours of the quantum torques (37) and (38) and thus of the geometry of the background associated with the quantum entropy: in the extreme condition of Fisher metric, the quantum information associated with the quantum potential (34) and thus the quantum-entropic length (36) generate the quantum torques (37) and (38) which, by exhibiting a precessional motion, determine the fact that the angular momentum vectors of each representative of the ensemble of a two qubits pair precess in unison or not and the characteristic probabilities of these concurrent or anti-concurrent motions.

Moreover, in the paper [40] Ramsak considered in particular the angle made by the angular momenta and the azimuthal angle made by the xy-plane projections of the momenta. The angular momenta $\vec{M}_1$ e $\vec{M}_2$ of the two qubits make an angle Φ in the bohmian space of hidden variables ξ satisfying the following relations

$$\langle \cos \Phi \rangle = \left\langle \frac{\vec{M}_1 \cdot \vec{M}_2}{\left|\vec{M}_1\right| \left|\vec{M}_2\right|} \right\rangle, \tag{43}$$

$$(\Delta \cos \Phi)^2 = \left\langle \cos^2 \Phi \right\rangle - \langle \cos \Phi \rangle^2 . \tag{44}$$

For $\vartheta = \pi/2$ one obtains the exact relation

$$\langle \cos \Phi \rangle = \frac{1}{3} (2 \cos \varphi - 1) \tag{45}$$

identical to the expression of standard quantum mechanics. Equation (45) implies that a perfect antiparallel alignment holds for the singlet state, while momenta for the triplet state, $\varphi = 0$, are only partially aligned, $\langle \cos \Phi \rangle = \frac{1}{3}$. As regards qubit pairs with vanishing total z-axis projection of spin, the azimuthal angle made by the xy-plane projections of the momenta is

$$\langle \cos \phi \rangle = \left\langle \frac{M_{1x} M_{2x} + M_{1y} M_{2y}}{\sqrt{\left(M_{1x}^2 + M_{1y}^2\right) \left(M_{2x}^2 + M_{2y}^2\right)}} \right\rangle \tag{46}$$

and the corresponding variance is

$$(\Delta \cos \phi)^2 = \left\langle \cos^2 \phi \right\rangle - \langle \cos \phi \rangle^2 . \tag{47}$$

Taking into account the guiding wave function (30) of the entangled system, Ramsak found that a finite φ represents only a shift of one of azimuthal angles of each member of the ensemble $\beta_2 \to \beta_2 + \varphi$, which results in the identity $\langle \phi \rangle = \varphi$ and the decoupling $\langle \cos \phi \rangle = C_B \cos \varphi$, where C_B is a function of ϑ only. An analogous result was obtained for $\langle \sin \phi \rangle = C_B \sin \varphi$ and thus $\langle \cos (\phi - \varphi) \rangle = C_B$ and $\langle \sin (\phi - \varphi) \rangle = 0$. A higher degree of entanglement can be visualized as a highly correlated distribution of angular momenta making azimuthal angles difference close to φ, with suppressed fluctuations for progressively increasing entanglement. In the geometrodynamic entropic approach here suggested, on the basis of equations (37) and (38) also the angle made by the angular momenta and the azimuthal angle made by the xy-plane projections of the momenta (as well as their consequences as regards the description of the entanglement) can be seen as a result of the quantum torques and thus of the deformation of the background associated with the informational lines of the quantum potential W_{k_1} and W_{k_2}, in other words of the quantum entropy for the two qubits (40). Ramsak's treatment in [40] of the entanglement properties of a qubit pair in the framework of Bohm's interpretation shows furthermore that the Bohmian analogue of Bell's inequalities, expressed in terms of Bohmian spin-spin correlators, is for fully entangled states identical to the standard quantum mechanics counterpart. In the picture proposed by Ramsak, the source of non-locality lies in the quantum potential which generates an instant coupling between the angular momenta of entangled qubit pairs. In fully entangled states the angular momenta precess in a particular manner forming a constant relative azimuthal angle φ which is the origin of a specific form of both, the standard quantum mechanical and the Bohmian correlators,

$$B\left(\vec{a},\vec{b}\right)=3\left\langle\frac{\left(\vec{a}\cdot\vec{M}_1\right)\left(\vec{b}\cdot\vec{M}_2\right)}{\left|\vec{M}_1\right|\left|\vec{M}_2\right|}\right\rangle=\left(a_xb_x+a_yb_y\right)B_x\cos\varphi+\left(a_yb_x+a_xb_y\right)B_x\sin\varphi-a_zb_zB_z \tag{48}$$

leading to the violation of the Bell inequalities. In the geometrodynamic entropic approach to quantum entanglement based on equations (33)-(38), since the quantum potential (34) corresponds with the deformation of the geometry described by the quantum-entropic length (36) and is produced by the quantum entropy, and the quantum information associated with the quantum potential (34) and thus the quantum-entropic length (36) generate the quantum torques (37) and (38) which determine the coupling of the angular momenta of entangled qubit pairs, the real, ultimate source of non-locality (and thus of the Bohmian counterpart of Bell's inequalities) is the deformation of the geometry of the background described by the quantum entropy and therefore the ultimate entities that determine the quantum information of the non-locality are the quantum torques (37) and (38). In synthesis, one can say that, on the basis of the geometrodynamic entropic approach to quantum entanglement based on equations (33)-(38), the ultimate source, the ultimate visiting card of quantum information is represented by the quantum torques (37) and (38) corresponding with the deformation of the geometry described by the quantum entropy.

4. About the Berry Phase in the Geometrodynamic Entropic Approach to Quantum Entanglement

A key approach of quantum computation research is to use quantum geometric phases (that is, the effects of moving a set of quantum parameters around a curved parameter space) to implement quantum gates that manipulate states of physical qubits [41-44]. Such gates would be the quantum computing equivalent of the logic gates found on today's microchips. The idea of using geometric phase is known as holonomic or geometric quantum computation, and has become one of the key approaches to achieving quantum computation that is resilient against errors. In 1999, Zanardi and Rasetti [44] laid the theoretical foundations of holonomic quantum computation by showing that any quantum circuit can be generated by using suitable Hamiltonians that depend on experimentally controllable parameters, such as those related to the manipulation of a bosonic mode in a quantum optical system [45]. In 2000 Jones, Vedral, Eckert and Castagnoli [46] demonstrated experimentally a quantum gate based on geometric phase that was able to entangle a pair of nuclear spins in a nuclear magnetic resonance setup. This experiment provided the first explicit example of geometric quantum computation demonstrating a controlled phase shift gate in which a conditional Berry phase is implemented and helped to boost the interest in this field.

In the paper "Berry-like phase and gauge field in quantum computing", Bruno, Capolupo, Kak, Raimondo and Vitiello showed that a gauge structure and a Berry-like geometric phase characterizes the initialization process of a qubit in quantum computing [47]. In this paper, these authors showed that, as regards the Berry-like phase within the qubit state, the fluctuating phases may be considered to be gauge-like degrees of freedom. Bruno's, Capolupo's, Kak's, Raimondo's and Vitiello's approach suggests a shift of paradigm in the study of quantum computing: typical problems arising in the analysis of qubit states, usually studied in the framework of quantum mechanics, are framed in the formalism of gauge theories. In such perspective, global geometric (Berry-like phase) characterization of the system and its gauge invariant behaviour are described as properties emerging from the collective behaviour of fluctuating quantum phases of qubit states.

Taking into account Bruno's, Capolupo's, Kak's, Raimondo's and Vitiello's results, the Berry-like phase involved in the qubit time evolution as regards the entangled state (11) for the n-cycle case is

$$\beta_\phi^{(n)} = -\frac{2\pi\omega_1}{\varphi} + \int_0^{nT} \langle \phi(t)| \, i\frac{\partial}{\partial t} \, |\phi(t)\rangle \, dt = 2\pi n \sin^2 \frac{\vartheta}{2} \tag{49}$$

where T is the period of oscillation,

$$|\phi(t)\rangle = e^{-i\omega_1 t}\left(\cos\frac{\vartheta}{2}\,|\uparrow\downarrow\rangle + e^{i\varphi}\sin\frac{\vartheta}{2}\,|\downarrow\uparrow\rangle\right), \tag{50}$$

$$\varphi = \omega_1 - \omega_2, \tag{51}$$

ω_1 and ω_2 being the eigenvalues (having the dimensions of frequencies) of the generator of phase transformation

$$H = \omega_1\,|\uparrow\downarrow\rangle + \omega_2\,|\downarrow\uparrow\rangle\,. \tag{52}$$

The Berry-like phase turns out to be physically equivalent to the Anandan-Aharonov invariant

$$s = 2\pi n \sin\vartheta \tag{53}$$

representing the distance between qubit evolution states, as measured by the Fubini-Study metric, in the projective Hilbert space (which is the set of rays of the Hilbert space into consideration). Equation (49) can also be rewritten in the following convenient form

$$\beta_\phi^{(n)} = \int_0^{nT} \left\langle \tilde{\phi}(t)\right| i\frac{\partial}{\partial t}\left|\tilde{\phi}(t)\right\rangle dt = 2\pi n \sin^2\frac{\vartheta}{2} \tag{54}$$

where

$$\left|\tilde{\phi}(t)\right\rangle = e^{-if(t)}\left(\cos\frac{\vartheta}{2}\,|\uparrow\downarrow\rangle + e^{i\varphi}\sin\frac{\vartheta}{2}\,|\downarrow\uparrow\rangle\right), \tag{55}$$

$f(t) = f(0) - \omega_1 t$, where f(0) is a constant.

Now, in the geometrodynamic entropic approach to quantum entanglement illustrated in chapter 3, the ultimate source of quantum information, namely the quantum torques

(37) and (38) derive from the deformation of the geometry associated with the quantum entropy and expressed by the quantum-entropic length (36). As a consequence, according to the authors it becomes legitimate to assume that the geometric invariant representing the distance between qubit evolution states, and thus the Berry-like phase, corresponds with the quantum entropic-length (36). Therefore one has

$$\beta_{\phi}^{(n)} = \int_0^{nT} \left\langle \tilde{\phi}(t) \right| i\frac{\partial}{\partial t} \left| \tilde{\phi}(t) \right\rangle dt = 2\pi n \sin^2 \frac{\vartheta}{2} = \frac{1}{\sqrt{\frac{1}{2I\hbar^2}\left(\frac{2}{W_1}\frac{\partial^2 W_1}{\partial x_{i_1}\partial x_{j_1}} - \frac{1}{W_1^2}\frac{\partial W_1}{\partial x_{i_1}}\frac{\partial W_1}{\partial x_{j_1}} + \frac{2}{W_2}\frac{\partial^2 W_2}{\partial x_{i_2}\partial x_{j_2}} - \frac{1}{W_2^2}\frac{\partial W_2}{\partial x_{i_2}}\frac{\partial W_2}{\partial x_{j_2}} \right)}} \tag{56}$$

which can be considered the definition of the Berry-like phase in the geometrodynamic entropic approach proposed by the authors. According to equation (56), one can say that the ultimate sources determining the Berry-like phase involved in the qubit time evolution as regards the entangled state (11) are the functions W_{k_1} and W_{k_2} defining the number of microstates of the two qubits and thus the quantum entropy (40). In other words, one can say that the Berry-like phase characterization of the system into consideration can be seen as a property which emerges from a more fundamental deformation of the geometrical properties of the background associated with the quantum entropy (40) of the system. We define the quantity (56) as the geometrodynamic entropic Berry-like phase regarding the two qubit system in the state (11).

Moreover, in analogy with Bruno's, Capolupo's, Kak's, Raimondo's and Vitiello's approach, one can consider $|\phi(t)\rangle \to e^{-if(t)} |\phi(t)\rangle = \left|\tilde{\phi}(t)\right\rangle$ as a local gauge transformation of the state $|\phi(t)\rangle$. On the basis of equation (55), one can observe that time evolution only affects the $|\downarrow\uparrow\rangle$ component of the state $\left|\tilde{\phi}(t)\right\rangle$. The gauge transformation acts as a "filter" freezing out (or compensating) time evolution of the $|\uparrow\downarrow\rangle$ state component, so that one obtains

$$\begin{aligned} i\hbar\frac{\partial}{\partial t}\left|\tilde{\phi}(t)\right\rangle &= -\varphi e^{-if(0)} e^{i\varphi} \sin\frac{\vartheta}{2} |\downarrow\uparrow\rangle = (H-\omega_1) e^{-if(0)} \left(\cos\frac{\vartheta}{2}|\uparrow\downarrow\rangle + e^{i\varphi}\sin\frac{\vartheta}{2}|\downarrow\uparrow\rangle \right) \\ &= (H-\omega_1)\left|\tilde{\phi}(t)\right\rangle \end{aligned} \tag{57}$$

Now, considering the state $|\phi(t)\rangle$ given by equation (50) and the state

$$|\psi(t)\rangle = e^{-i\omega_1 t}\left(-\sin\frac{\vartheta}{2}|\uparrow\downarrow\rangle + e^{i\varphi}\cos\frac{\vartheta}{2}|\downarrow\uparrow\rangle \right) \tag{58}$$

one has

$$H|\phi(t)\rangle = \omega_{\phi\phi}|\phi(t)\rangle + \omega_{\phi\psi}|\psi(t)\rangle, \tag{59}$$

$$H|\psi(t)\rangle = \omega_{\psi\psi}|\psi(t)\rangle + \omega_{\phi\psi}|\phi(t)\rangle \tag{60}$$

where

$$\omega_{\phi\phi} = \omega_1\cos^2\frac{\vartheta}{2} + \omega_2\sin^2\frac{\vartheta}{2} = \langle\phi(t)|H|\phi(t)\rangle \tag{61}$$

$$\omega_{\psi\psi} = \omega_1 \sin^2 \frac{\vartheta}{2} + \omega_2 \cos^2 \frac{\vartheta}{2} = \langle \psi(t) | H | \psi(t) \rangle \tag{62}$$

$$\omega_{\phi\psi} = \frac{1}{2} (\omega_2 - \omega_1) \sin \vartheta = \langle \psi(t) | H | \phi(t) \rangle . \tag{63}$$

Equations (59) and (60) can be written in the following compact form

$$(H - \omega_{\phi\psi} (|\phi(t)\rangle \langle \psi(t)| + |\psi(t)\rangle \langle \phi(t)|)) |\zeta(t)\rangle = \omega_d |\zeta(t)\rangle \tag{64}$$

where $\omega_d = diag(\omega_{\phi\phi}, \omega_{\psi\psi})$. In equation (64) the operator

$$F = (H - \omega_{\phi\psi} (|\phi(t)\rangle \langle \psi(t)| + |\psi(t)\rangle \langle \phi(t)|)) \tag{65}$$

may be interpreted as the free energy operator and may also be expressed as

$$F = \lambda S_i \tag{66}$$

where λ is the temperature of the entangled qubit pair and S_i are the superposed Boltzmann entropies given by equation (40).

Another possible manner to generate a Berry phase has been shown theoretically in flux qubits [48] by applying an external field. As a consequence, varying the interaction strengths and applying an external field may cause both the entanglement [49, 50] and the phase dynamics [51] of the subsystems to change. In this regard, in the recent paper "Relation between quantum entanglement and the Berry phase in systems of two interacting qubits", Chen, Cho and Su investigated quantum entanglement and the Berry phase in systems of two interacting qubits, where the interaction has the form of an XXZ-type exchange interaction, which makes the qubits entangled [52]. By considering a slowly rotating external field within the adiabatic theorem, these authors found analytic expressions for the eigenvalues and eigenfunctions of the system in terms of the interaction parameters and external fields and studied the effects of interactions between the Berry phases and entanglements for the eigenstates of the system. For the interaction between two qubits Chen, Cho and Su considered a XXZ Hamiltonian of the form

$$H = H_x + H_z + H_B = \begin{pmatrix} 2B_0 \cos\theta + J_z & B_0 \sin\theta e^{-i\varphi} & B_0 \sin\theta e^{-i\varphi} & 0 \\ B_0 \sin\theta e^{i\varphi} & -J_z & 2J_x & B_0 \sin\theta e^{-i\varphi} \\ B_0 \sin\theta e^{i\varphi} & 2J_x & -J_z & B_0 \sin\theta e^{-i\varphi} \\ 0 & B_0 \sin\theta e^{i\varphi} & B_0 \sin\theta e^{i\varphi} & -2B_0 \cos\theta + J_z \end{pmatrix} \tag{67}$$

where $H_x = J_x (S_{1x} S_{2x} + S_{1y} S_{2y})$, $H_z = J_z S_{1z} S_{2z}$ and $H_B = \mu \left(\vec{S}_1 + \vec{S}_2 \right) \cdot \vec{B}(t)$, μ being the gyromagnetic ratio, J_x and J_z the exchange interaction strengths between the qubits, $\vec{B}(t) = B\hat{n}(t)$ a slowly rotating external field with the unit vector $\hat{n}(t) = (\sin\theta \cos\phi(t), \sin\theta \sin\phi(t), \cos\theta)$. After computing the eigenvalues and eigenstates of the Hamiltonian (67), Chen, Cho and Su showed that during the adiabatic and cyclic evolution with an initial eigenstate $|\psi_n(0)\rangle$, there are no quantum transitions among any of the states, and the system returns back to the initial state with additional phases. In this picture, the geometrical background of the system determines a Berry phase of the form

$$\beta_n = \int_0^{2\pi} d\phi \, \langle\psi_n| \, i\partial_\phi \, |\psi_n\rangle = \frac{16\pi B_0^3 \varepsilon_n \sin\theta \sin 2\theta}{\varepsilon_n^4 - 2(1+3\cos 2\theta) B_0^2 \varepsilon_n^2 + 16 B_0^4 \cos^2\theta} \tag{68}$$

where

$$\varepsilon_n = 2\sqrt{Q}\cos\left(\frac{P+2n\pi}{3}\right) + \frac{2}{3}J, \tag{69}$$

$$Q = 4\left(J^2 + 3B_0^2\right)/9, \tag{70}$$

$$P = \cos^{-1}\left(\frac{R}{\sqrt{Q^3}}\right), \tag{71}$$

$$R = 4J\left(2J^2 + 9B_0^2\left(1 - 3\cos^2\theta\right)\right)/27. \tag{72}$$

According to Chen's, Cho's and Su's results, the Berry phase turns out to be zero when an eigenstate becomes a maximally entangled state and also un-entangled eigenstates could capture a finite value of Berry phase. Therefore, in Chen's, Cho's and Su's approach, the anisotropy of the exchange interaction determines the behaviour of the relation between the Berry phase and entanglement for two interacting qubits.

In the geometrodynamic entropic approach suggested in this article, the Berry phase (68) characterizing systems of interacting qubits where the interaction Hamiltonian has the form (67) may be written as:

$$\begin{aligned} \beta_n &= \frac{16\pi B_0^3 \varepsilon_n \sin\theta \sin 2\theta}{\varepsilon_n^4 - 2(1+3\cos 2\theta) B_0^2 \varepsilon_n^2 + 16 B_0^4 \cos^2\theta} \\ &= \frac{1}{\sqrt{\frac{1}{2I\hbar^2}\left(\frac{2}{W_1}\frac{\partial^2 W_1}{\partial x_{i_1}\partial x_{j_1}} - \frac{1}{W_1^2}\frac{\partial W_1}{\partial x_{i_1}}\frac{\partial W_1}{\partial x_{j_1}} + \frac{2}{W_2}\frac{\partial^2 W_2}{\partial x_{i_2}\partial x_{j_2}} - \frac{1}{W_2^2}\frac{\partial W_2}{\partial x_{i_2}}\frac{\partial W_2}{\partial x_{j_2}}\right)}}. \end{aligned} \tag{73}$$

According to equation (73), one can say that the interaction between the two qubits determines a deformation of the geometrical properties of the background which corresponds with the functions W_{k_1} and W_{k_2} defining the number of microstates of the two qubits and thus with the quantum entropy of the system (40). Equation (73) allows us to provide a geometrical re-reading to Chen's, Cho's and Su's model of quantum entanglement and Berry phase in which the external field and the exchange interaction strengths describing the interaction between the two qubits are associated with a fundamental deformation of the background expressed by the quantum entropy. We can define the Berry phase (73) as the geometrodynamic entropic Berry phase regarding two interacting qubits with external field and Hamiltonian (67).

5. About the Perspectives of the Geometrodynamic Entropic Berry-Like Phase in Quantum Computing

Geometric quantum computation employs one-dimensional geometric phase factors to achieve universal sets of quantum gates. One significant way to achieve quantum computation is based on the role of the Berry phase, which occurs in situations like the Aharonov-Bohm setup where a charged particle confined to a box acquires a geometric phase while slowly taking the box around a magnetic flux. The Berry phase can be thought of as an adiabatic quantum holonomy restricted to a one-dimensional energy eigenspace. Berry phases may be used for quantum computation by encoding the logical states in nondegenerate energy levels, such as in the spin-up and spin-down states of a spin 1/2 particle in a magnetic field. When this field rotates slowly around a loop, the spin states will pick up Berry phases of magnitude given by half the enclosed solid angle and of opposite sign, which defines an adiabatic geometric phase-shift gate acting on the two spin states. But these states also pick up different dynamical phases (due to the Zeeman splitting caused by the magnetic field interacting with the spins), which one needs to compensate for. Jones, Vedral, Ekert and Castagnoli [46] proposed to remove these dynamical phases by a clever sequence of radio-frequency fields interrupted by suitable p-pulses (which swap the spin-up and spin-down states) applied to a pair of coupled nuclear spins in a nuclear magnetic resonance setup.

In our approach, the starting-point to throw new light on geometric quantum computation is represented by the geometrodynamic entropic Berry-like phase (56) or by the geometrodynamic entropic Berry phase regarding two interacting qubits with external field (73). According to equations (56) and (73) one can say that the deformation of the geometrical properties of the background play a crucial role in the analysis and characterization of quantum computing. In particular, the so-called initialization problem in qubit preparation turns out to be strictly related with the geometrodynamic Berry phases (56) and (73).

Soon after proposals for quantum computing were advanced, it was argued that problems of decoherence and noise precluded physical implementation of quantum computers [53, 54]. The computing system was not to interact with the environment in order to avoid its decoherence. At the same time, however, the system was required to be strongly and precisely interacting with the control circuitry. As regards the question of the initialization of the quantum state [55-57], one might assume a conceptual scheme (as for example the one adopted in the reference [58]) where each noisy gate is replaced by an ideal gate by the use of error correction circuitry, but such an idea is impractical. Even if it is assumed that the errors are below a certain threshold, correction in the quantum context can work only if it is related to a known state; in contrast, error correction in classical computation works for unknown states. In fact, on the basis of Kak's treatment in [59], one can say that certain gate faults could not be corrected.

It was argued that for quantum computing verifiability of the computation is not always possible even if one were to ignore questions of decoherence and noise. In the paper [60], Kak discussed in detail the problem of quantum gate testing suggesting that the complexity of this testing will increase exponentially as the space of the probability amplitudes is exponentially related to the number of observational variables. According to this view, if quantum gates cannot be effectively tested, quantum computers cannot be run effectively. Here, taking into account Kak's discussion, we can argue that the complexity of the quantum gate testing is directly related with the functions W_{k_1} and W_{k_2} defining the number of microstates of the two qubits and thus the quantum entropy (40) of the system. In our approach, one can say that it is not possible to completely observe, algorithmically test and control quantum gates as a consequence of the deformation of the geometrical properties of the background associated with the microstates of the two qubits and thus with the quantum entropy of the system.

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