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Quantized Interest Rate at The Money for American Options

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Abstract: In this work, we use the Bachelier stochastic differential equation as our model for the stock price movement. We assume an investor has entered into an American call option contract such an investor would want the stock price to end up above K (strike price) in order to get a positive expected future payoff. We assume also the stock price to be below K and moving it way up into the deep in the money state a. Then we use martingale, supermartingale, and Markov and Ito calculus to obtain a Bachelier-type of the Black-Scholes-Merton equation which we hedge to obtain by comparison the time independent Schroedinger equation in Quantum Mechanics. Finally, we solve the time independent Schroedinger equation for the interest rate and the expected future payoff of the stock holder at the money, $X(t) = K$.

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1. Introduction

Much finance research in early 1970s concentrated on valuations of options assuming a specific model for the underlying asset, in this case stock. It is important to mention here that Black-Scholes and Merton published their work. Some of the ideas contained on those papers dated back to the early part of the 20^{th} century, such as the representation of the stock movement as a random process. More than a century ago, the description of stock price movement began with Bachelier [4] when He wrote his dissertation by assuming the stock price movement to behave like a random walk. His theoretical model was based on expectations; stock price movements are random but what happens on average?.

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Between Bachelier and Black-Scholes-Merton nothing much happened in the theory of pricing options. Bachelier's equity model using asset price changes drawn from a normal distribution was modified in 1950s to normally distributed asset price returns (asset price changes divided by the asset price). This was called the lognormal distribution for asset price changes. But, one needs to recognize that Black-Scholes-Merton made some insights that were to fundamentally change the universe of finance and in a very short time to make finance a serious mathematical subject with applications from classical physics to Quantum mechanics and even to Quantum Field Theory. In finance, the process of eliminating portfolio sensitivity to changes in the underlying is called delta hedging or dynamic hedging; hedging is considered to be the most general form for reduction of variability or risk. Black-Scholes in their delta hedging made assumptions on how to eliminate variability or risk by constructing a simple portfolio. Since then, several attempts have been made in describing the stock price movement by introducing different models. However in this work, I am not reinventing the wheel I am going instead to use a well known and accepted model within research to describe the stock price movement for an expected future payoff of an investor who has entered into an American call option contract with maturity.

The assumptions made by Black-Scholes-Merton are the following, they assumed the stock price movement to be described in terms of constant interest (drift term) and constant volatility (diffusion) terms. The other interesting assumption they made was that information does not really affect a given stock price movement and all players within the market possess the same information distribution about the market. Any non-uniform distribution of information plays no role at all and all the non uniform information distribution is quickly taken care of by the market.

After reading John Hull's book [1], I began questioning the assumptions made by Black-Scholes in their pricing model. Luckily in early 2004, I began doing research for a Hedge Fund in Manhattan working with an experienced trader. The first week at the job as a physicist, I was eager and curious to check some of the assumptions made by Black-Scholes-Merton by using real time market data. This was possible to be accomplished since I was exposed to a huge amount of real time trading data from the S&P 500 companies and Dow Jones just to give a few [18]. I Looked at historical data for analysis. I began extracting volatilities for different companies on a short and long period of time. What was discovered was that volatilities of all the companies were changing with time, and similar results were observed for interest rates or returns on given stock. It is also important to mention that most of the models for stock or asset pricing do take into account the randomness of the market. This assumption would make a few questions arise and one of the questions is. I s the market really a random process, one can say yes if you're coming from an academic department and one can say no if you're a trader trading on Wall Street. By the way, the majority of the traders I have interacted with do not believe that the market behaves as a random process. Therefore, after my experience as a quant at the Hedge Fund I do not either believe that the market is a random process. But if one is coming from an academic department without knowledge of the moving forces of the markets then the assumptions given above can be made.

Above all, the Black-Scholes-Merton equation is a parabolic partial differential equation. Mathematically, it is of the same type as the well studied heat equation in fundamental physics. Indeed the heat or diffusion equation has been around for almost two centuries. And suddenly finance becomes a field of interest to mathematicians and physicists, and there was more to it than simple compound interest. On the other hand it keeps theoretical physicists and mathematicians busy in finding out ways how to model finance and economics problems.

Unlike in finance, in physics once the position of the particle is given one can solve for the velocity and the acceleration of the particle and in general one can model the motion of the particle in space and time.

On the other hand, the world of finance is far removed from the physical world and there is no reason to believe that there should be any immutable laws or principles like there are in the world of physics.

From a research stand point in finance, one can make the assumption that the change of the stock price is proportional to some drift term that could influence the stock movement or further the change of the stock movement in time is proportional to the stock price multiplied by the same drift in order to get an exponential growth all the time. The exponential growth is very important for an investor investing in a company. Then, based on this assumption the expected return of a company is going to grow all the time and will never reach a state of bankruptcy [12]. In the article [12] their quantum derivation of the Black-Scholes type is based on the fact that companies do never reach a state of bankruptcy. This would be deterministic with a guaranteed probability of some return in time and no matter how long it takes there is always growth without any risk.

It is well known that the Black-Scholes-Merton model is the most accepted and used model on Wall Street despite the model is far from being perfect and is inconsistent with the realty of the market. And Because the model is far from being perfect researchers and practitioners on Wall Street are working hard to find other ways in order to improve the well known model. And also very often these improvements are completely inconsistent with the rest of the theory and lack some basic principles of finance. One of the questions I asked the trader I was working with was that how come companies get removed from the S&P 500 listing, He answered by saying for poor performance companies are always replaced with other high performing companies. So, a company's growth or return can change over time and this could lead into bankruptcy (case of Lehman Brothers, Bear Stearns Companies).

Now if we look at it from the researcher's stand point from a given institution that does not possess real time financial data or enough information and knowledge about the market can always introduce randomness when modeling the market price behavior. Now, in order to be on the safe side of the universe of finance He will maintain the change of the stock price to be proportional to the deterministic (no randomness) and non deterministic (randomness included) parts multiplied by the stock price itself. Hence, since the black-Scholes-Merton (Bachelier) model does not include information in addition to the assumptions gives above then; the model does have some serious inconsistencies that need to be addressed. Despite of all the assumptions, I still believe that the field of finance is interesting and available for all kind of phenomenological applications from Classical physics to quantum physics including quantum field theory. At the same time, we disagree with [12] that the quantum nature of price dynamics can really occur in the same way as in physics. All these theories on pricing are all phenomenological including our work herein.

Recently, several power law tails have been observed in recent years for the total distribution of market indices, on returns for different companies and on commodities prices as well. A power law tail with an exponent $\alpha \sim 3$ has been proposed recently [14]. Several authors have linked this asymptotic power law to the trading behavior of large financial institutions and to big investors on Wall Street [15]. I agree with [15] and I would say that these power law tails come from the fact that market movers and market shakers do possess different degrees of information about the state of the market and this non uniformity of the distribution of information contradicts the Black-Scholes-Merton's pricing model which does not take information into account. We are working on an article that is going to describe some links between information and price movement and this is not within the scope of this present work. Also, several authors [2, 5, 6, and 7] have solved several problems in stochastic optimization using martingale theory. They have investigated stochastic optimization problems by assuming both the Black-Scholes (multiplicative) and Bachelier (additive) models as models for stock price movements. We will use as our model for stock price movement the Bachelier model also known as the additive model [4] to find the expected future payoff of a stock holder who has entered into an America call option contract.

In this work, we divide our task in to two mains parts: for the first part, we will look at the expected future payoff at the money ($X = K$) with zero expected future payoffs for a stock holder who has entered in to an American call option contract. For the second part, we will call at the money point a singularity point with zero expected future payoff and solve for interest rate, then in sections 3 and 4 we use stopping time or Markov time and martingale and supermartingale theories to obtain an Ito differential equation. Ito calculus will be used to obtain a Bachelier-type of the Black-Scholes pricing equation, eventually this equation will be hedged to obtain an equation that is similar to the time independent Schoredinger equation in Quantum mechanics. The hedging concept is explained accordingly in order to understand the connection between the two equations. The results are summarized in section 5.

2. Bachelier Model, American Call and Put Options and the Expected Future Payoff

Given below is the equation for the stock price movement:

$$X(t,\omega) = x + rt + \sigma W(t) \tag{1}$$

Where at $t = 0, X(t = 0) = x$ is the initial stock price (present value of the stock) and equation (1) is called the Bachelier Brownian model with drift $\mu = r$ sometimes called the return on a given stock and volatilityσ we have assumed both$r$ and σ are constant. Both parameters do not depend on time or on the stock price. Equation (1) given above satisfies the following stochastic Ito differential equation (additive model) with constant parameters.

$$dX(t,\omega) = \mu dt + \sigma dW(t) \tag{2}$$

$X(t,\omega)$ is a stochastic process, $t \in T, \omega \in \Omega$ where (Ω, F, P) is a probability space on which there is a Wiener process, $W(t,\omega)$ is measurable with respect to a family F_t of σSubfields F is called a diffusion process if it satisfies the stochastic Ito differential equation (2) given above. We would call the first and second terms in equation (2) to be the deterministic (no-randomness) and non-deterministic (randomness included) terms respectively.

Traders on Wall Street prefer to price options using the Black-Scholes model (multiplicative model) for one simple reason that the change of the stock price movement is proportional to the stock price itself which gives an exponential growth all the time. In the Black-Scholes model both the drift and volatility terms were assumed to be constant which supports our assumptions in this work.

We have done simulations to check these two models (Black-Sholes and the Bachelier models) and what we found was that in contrast to the Black-Scholes model the Bachelier model can have negative prices while in reality stock prices are non negative [18]. However, in this work we are not interested in understanding single stock prices but we are interested in the probability distributions of stocks so we will treat the total expected future payoff as a measurable quantum variable.

Options are known within the financial market as a type of derivative, which are given to stock holders in order to hedge their positions against risky fluctuations of the stock price. They are divided into two categories: call and put options.

A call option is a contract that gives the right to the holder to buy the underlying asset by a certain date for a certain price (strike price) in the future. An investor holding such an option would wish the stock price to go higher than the strike price (or exercise price). Then the expected future payoff of the call option is given by:

$$e^{rt} max(X(t) - K, 0), X(t) > K \tag{3}$$

Since the present value of a price is negatively discounted in terms of the future price [1] that is:

$$PV = FVe^{-rt}$$

where PV and FV stand for present and future values of prices respectively at a given time and $r$is the risk free interest rate.

Then one can compute the future price from the present value equation given above that is:

$$FV = PVe^{rt}$$

$Wewantedtoshowthatboththefuturevalueandthefutureexpectedpayoffgivenbyequation(3)arepositivelydiscountedwithrespecttotheexponen$
future and the present value equations came from the fact that if one assumes the change of the price to be proportional to the price itself this would lead to an exponential growth that is given by the first order differential equation below:

$$d\phi = \phi\left(\mu dt\right) \Leftrightarrow \frac{d\phi}{\phi} = \mu dt$$

$$\Rightarrow \int \tfrac{d\phi}{\phi} = \mu \int^{d} t \Leftrightarrow \phi\left(t\right) = \phi_0\left(e^{\mu t}\right)$$

We have computed the future price that is positively discounted in terms of the risk free interest rate. It is important to mention here that based on the future value price that is discounted positively we were able to write the expected future payoff of the call option positively discounted. The present or future value of the price is sometimes called the time value of money [13, 1]. The exponential growth derived above with the assumption that interest rate does not change with time and other market parameters in this case $\mu = r$ (μ is sometimes called the return) is the time value of money formula.

A put option is also a contract that gives the right to the holder to sell the underlying asset by a certain date for a certain price (or strike price). An investor holding such an option would wish the stock price to go below the strike price (or exercise price). The expected future payoff of the put option is given by:

$$e^{rt} max\left(K - X\left(t\right), 0\right), X\left(t\right) < K \tag{4}$$

where$X\left(t, \omega\right)$ is given by the Bachelier model (1) and K is the strike price or exercise price and (3) and (4) are discounted with the risk free interest rate. In a non arbitrage situation $\mu = r$, is the risk free interest rate in the Bachelier model. In a non arbitrage situation it has been believed that market makers traders or investors do not have enough information about the market behavior that could affect its prices.

The put option described by equation (4) is not going to be used in this work even though we could have used it but we rather decided to use the call option case described by equation (3). However, we are working on options pricing models which involve both the call and put options with the assumption that interest rate and volatility do not change with time and with other market parameters. Several authors have assumed interest rates and volatility models in which both interest rate and volatility are assumed to be stochastic differential equations. Interest rate and volatility models are not within the scope of this article.

There are a few parameters that affect the price of options when they are issued to stock holders and these parameters are: the stock price (described with the Bachelier model), the risk free interest rate$r$, the strike price or exercise price K, the volatility of the stockσ , time to maturity (expiration date of the option) T and other dividends during the life time of the option. We assume here that there are no dividends being paid to the stock holder during the entire time of the call option [1].

In this work, we will examine the American option type. An American option in general is an option that can be exercised at any given time before the expiration date of the option. Since an option is issued to the stock holder in order to hedge his or her position against risky fluctuations of the stock then he can get either into an American call or an American put options contracts. Our main assumption in this work is that the stock holder has entered into American call option contract instead of an American put option contract.

An American call option could be deep in the money when the expected payoff is always positive given by equation (3) or out of the money when the expected payoff is always negative which is given by equation (4). It could also be at the money when the expected payoff is zero which is given by the following equation:

$$e^{rt}max\left(X\left(t\right)-K,0\right)=0\Rightarrow X\left(t=\tau\right)=K \tag{5}$$

According to equation (5) given above, the expected future payoff of the stock holder at the money is zero. In order to achieve this goal we will introduce the Markov stopping time at the money when the stock price described by the Bachelier stochastic Ito differential equation is constant and equals to the strike price or exercise price at the stopping timeτthat is:

$$V\left(X\left(t=\tau\right)\right)=e^{rt}max\left(X\left(t=\tau\right)-K,0\right)=0$$

$$\Rightarrow X\left(t=\tau\right)=K \tag{6}$$

Equation (6) is our main boundary value equation for the expected future payoff at the money. In finance, when your expected payoff is zero it means you have lost money at that given time on a specific given trade. Using Ito calculus and martingale theory we will derive a Bachelier's type pricing equation which we will hedge and eventually compare it to the non-relativistic time independent Schroedinger equation in quantum mechanics. In the next section we will introduce stopping time or Markov time, martingale and Ito calculus.

3. Stopping time or Markov time and Martingale

A random variable τ defined on the same sample space as a martingale or Markov process is a stopping time if for every time t, the event:

$$\{\tau\leq t\}\in F_t$$

What does this really mean; it means that for any time t one can tell whether or not $\tau\leq t$

That τhas occurred or not, by just knowing all the information up to time t one doesn't need to have any information about the future.

A stochastic process $X_n\left(t,\omega\right),n=0,1\ldots$ is called a martingale if the two conditions are met [9, 10]:

$$E\|X_n\left(t,\omega\right)\|<\infty \quad (i)$$

Since the expected payoff is defined in terms of the price movement described by the Bachelier model, from (i) we can write the following for the expected payoff:

$$E\|V_n(t,\omega)\| < \infty$$

$$E\|X_{n+1}(t,\omega)\,|X_{0........}\,X_n\| = X_n(t,\omega) \quad (ii)$$

From *(ii)* we can obtain the conditional expectation for the expected payoff:

$$E\|V_{n+1}(t,\omega)\,|V_{0........}\,V_n\| = V_n(t,\omega)$$

Let's try to understand the meaning of the two above expectations, (i)means that the expected price is always finite and *(ii)* means that the expected value of the future price given the present price is equal to the present value of the price. All information about the future price of the stock is already incorporated into the present value; it also means that there are no arbitrage opportunities.

Figure 1. below shows the three states that are: when the stock price is deep in the money (state a) with a positive expected payoff, the stock price is deep out of the money (state c) with a negative expected payoff and finally the stock price is at the money (state b) when the expected payoff is zero which corresponds to equation (6).

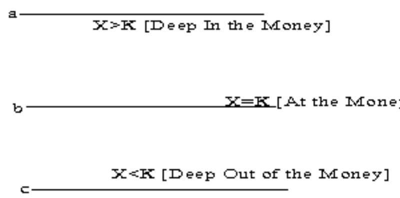

Fig. 1 $X = K$[At The Money], $X > K$[Deep In The Money] and $X < K$[Deep Out Of The Money]

In this article, we have assumed that the stock holder has entered into an American call option type of contract which means that he or she can exercise the option at any given time before the expiration date of the option. Hence, if the stock goes deep out of the money below state b, He or She is not going to exercise because the expected payoff is negative. The best scenario for the stock holder is to wait until the stock moves up deep in the money or somewhere above sate b then exercise with a positive payoff.

From Figure 1, when the stock moves let's say from deep out of the money state c toward the deep in the money state a , it needs to move through the state b where the expected payoff is zero while the stock holder is awaiting for the stock to move up deep in the money. We will investigate how market parameters such as the risk free interest rate, volatility of the stock behave with respect to one another while the stock holder is awaiting for the stock to move up well above state b.

4. Definition of Supermartingale

Since the expected payoff of the American call option deep in the money is a positive function using stopping time and martingale theories, we wish the stock price to reach the exercise price$K$ given by equation (6).

Now let's assume that there is a stochastic process [8, 3] $Y(t,\omega)$which is defined in terms of the expected payoff $V(X(t,\omega))$ discounted at the risk free interest rate via the exponential function in the following:

$$Y(X(t,\omega)) = V(X)e^{rt} \tag{7}$$

Equation (7) is nothing else but the expected future payoff of the stock holder discounted at the risk free interest rate via the exponent. In a risk free environment and for short term trades, we assume these rates $r$to be very small.

Using stopping time and martingale theories, we would want the expected future payoff of the American call option at the money state $b$to vanish.

If $\{\tau \le t\} \in F_t$, there is a stopping time at which $\tau = t'$where $\{t' < t\} \in F_t$.

This corresponds to the following expected future payoff at the money state b, where

$$V(X(t,\omega)) = E\|X(\tau) - K, 0\| \tag{8}$$

At $X(\tau) = K$ we want the *sup.* of equation (6) to vanish:

$$V(X(t,\omega)) = SupE\|X(\tau) - K, 0\| \,|(X(\tau) = K) = 0 \tag{9}$$

Then (7) becomes:

$$\Rightarrow Y(t,\omega) = SupE\|(X(\tau) - K, 0)e^{rt}\| \,|(X(\tau) = K) = 0$$

In order to take care of the singularity $(Y(t,\omega) \to 0)$ for$Y(t,\omega)$, that is the total expected future payoff of the stock holder we will need to use Ito calculus to obtain the Bachelier type of the Black-Scholes equation.

Now let's use Ito calculus and martingale to derive the Bachelier type of Black-Scholes equation for the expected future payoff $V(X(t,\omega))$. Eventually, we will compare the obtained equation to the time independent non-relativistic Schroedinger equation.

If $Y(t,\omega)$ is a local supermartingale then it does have the following differential equation:

Using the fact that the conditional expectation of the differential given the present value is given by equation (7) at certain time t is: $\|dY(t,\omega)\,|X(t) = x\,\|$, where the full differential of (7) is given below:

$$dY(t,\omega) = \frac{\partial Y}{\partial t}dt + \frac{\partial Y}{\partial x}dx + \frac{1}{2}\frac{\partial^2 Y}{\partial x^2}(dx)^2 + \ldots \tag{10}$$

The first term in the full differential of $Y(t,\omega)$corresponds to the derivative of equation (7) with respect to time and the second and third derivatives are with respect to the stock price. Then, one can write the conditional expectations:

$$E\|dY(t,\omega)\,|F_t\,\| = 0 \tag{11}$$

$$E\|dY(t,\omega)\,|F_t\,\| \le 0 \tag{12}$$

The expectation given by equation (12) sounds little confusing since it is a supermartingale. This is how supermartingale has been defined. In contrast to a submartingale which has an increasing expectation a supermartingale has a decreasing expectation given by (12). Submartingality is out of the scope of this work.

In order to obtain the Bachelier type of Black-Scholes equation using martingale and supermartingale one needs to substitute the full differential of $Y\,(t,\omega)$given by (10) into the conditional expectation described by equation (12):

$$E\| \frac{\partial Y}{\partial t}dt + \frac{\partial Y}{\partial x}dx + \frac{1}{2}\frac{\partial^2 Y}{\partial x^2}(dx)^2 + \ldots |F_t \,\| \le 0 \tag{13}$$

Using the Bachelier's stochastic differential equation given by equation (2) then we will obtain:

$$dX\,(t,\omega) = rdt + \sigma dW\,(t)$$
$$(dX)^2 = (rdt + \sigma dW\,(t))^2 = (rdt)^2 + (\sigma dW\,(t))^2 + 2rdtdW\,(t) \tag{14}$$

By differentiating equation (7) with respect to time and also by substituting stochastic differential equations given by equations (2) and (14) we will obtain the following expectation from equation (13):

$$E\left[rV\,(x)\,e^{rt}dt + \frac{\partial Y}{\partial x}\,(rdt + \sigma dW\,(t)\,) + \frac{1}{2}\frac{\partial^2 Y}{\partial x^2}(rdt + \sigma dW\,(t))^2 + \ldots |F_t \right] \le 0 \tag{15}$$

and using the Ito matrix given below:

$$\begin{pmatrix} dW & dt \\ 0 & 0 \end{pmatrix}\begin{pmatrix} dW & 0 \\ dt & 0 \end{pmatrix} = \begin{pmatrix} dW^2 & dtdW \\ dWdt & dt^2 \end{pmatrix} = \begin{pmatrix} dt & 0 \\ 0 & 0 \end{pmatrix}$$

We were able to get the following equalities according to Ito calculus:

$$dW^2 = dt, dWdt = dtdW = dt^2 = 0$$

Since $EdW\,(t) = 0$ from equation (13), then the expectation (15) becomes:

$$E\| \left(rV\,(x) + r\frac{\partial V\,(x)}{\partial x} + \frac{1}{2}\sigma^2\frac{\partial^2 V\,(x)}{\partial x^2} \right) e^{rt}dt\,|F_t \,\| \le 0 \tag{16}$$

If the multiplicative stochastic differential equation (Black-Scholes) and the additive stochastic differential equation are martingale then their drift terms (coefficients ofdt) of both equations are zero:

$$dX(t,\omega) = rdt + \sigma dW \quad r = 0$$

$dX(t,\omega) = \sigma dW$ (Bachelier model)

$$dX(t,\omega) = X(rdt + \sigma dW) \quad r = 0$$

$dX\left(t,\omega\right) = X\sigma dW$ (Black-Scholes)

The Black-Scholes model is not going to be used as our model for the stock price in this work however we are working on an article where the Black-Scholes model has been used as our model price movement.

Since a supermartingale is a martingale we will obtain from the expectation (16) the coefficient of dt to vanish.

$$rV + r\frac{\partial V}{\partial x} + \frac{1}{2}\sigma^2\frac{\partial^2 V}{dx^2} = 0 \tag{17}$$

Equation (17) is an ordinary differential equation with constant coefficients for the expected future payoff of the American call option where r is the interest rate and σ^2 is the volatility of the market. In traders language on Wall Street, from equation (17) the first and second derivatives of the expected future payoff of the American call option with respect to the price is called (delta) ? $= \frac{\partial V}{\partial x}$and (gamma) $\Gamma = \frac{\partial^2 V}{\partial x^2}$respectively [1].

Equation (17) is also called the classical formula for the Bachelier type of the Black-Scholes equation for the call option with a strike price (exercise price) K at any time before the expiration of the option. The beauty about equation (17) is that it can be hedged periodically, traders on Wall Street worry when $\frac{\partial V}{\partial x}$ is too high [1]. If one didn't work on all Wall Street it could be difficult to understand it but Physicists (I have seen it while I was there) called sometimes "Quants" who have been there know that hedging for a trader is very important in order to eliminate risk. We will explain hedging of delta or sometimes called delta-hedging or delta-neutral below.

In the limit when ? $= \frac{\partial V}{\partial x} \to 0$, the second term in equation (17) is going to vanish and this corresponds to delta hedging or delta neutral in finance. What is the meaning of delta hedging in finance; it means the price sensitivity of the expected payoff with respect to the underlying instrument. It is also a technique used by traders who on a daily basis trade options because trading options requires many transactions. We could also understand delta neutral as no matter how the price moves the expected payoff will not change in value and another better interpretation of the delta hedging is the probability that the expected payoff is going to end up in the money given by equation (3). Edward Nelson [17] was able to derive the time independent Schroedinger equation by assuming a classical force given by Newton's second law of motion acting on a particle with mass m which was assumed to perform a Markov process. The time independent Schroedinger equation derived by Edward Nelson when compared to equation (17) is identical and one can identify the volatility of the market σ^2as reciprocal of mass of the system. Hence, hedging equation (17) and treating it as a quantum equation we will obtain a one dimensional time independent Schroedinger equation with a Hamiltonian describing a free particle.

Let's find the solution to equation (17) that is the expected future payoff at the money state b, where $X\left(\tau\right) = K$ of the American call option. Hence, by delta-hedging (17) and comparing it to the time independent Schroedinger equation [11] for a free particle with no forces disturbing the particle from its state of equilibrium then we will have:

$$-\frac{}{2m}\frac{d^2}{dx^2}\psi = E\psi \Longleftrightarrow -\frac{1}{2}\sigma^2\frac{d^2}{dx^2}V = rV$$

$$\Rightarrow \frac{1}{2}\sigma^2 = \frac{}{2m} \Rightarrow \sigma^2 \cong \frac{1}{m} \tag{18}$$

One can rewrite the time independent Schroedinger equation given above in terms of the Hamiltonian H. The Hamiltonian here is absolutely time independent.

$$(H + E)\,|\psi> = 0$$
$$\Rightarrow (H + r)\,|V> = 0 \tag{19}$$

where $|V>$ is a measurable quantum state of the expected future payoff with a probability that the expected future payoff is going to be found in this quantum state, r is the risk free interest rate and H is the Hamiltonian describing the dynamics of the price movement within different markets. Markets are based on many different types of trades that are daily traded among movers and shakers of the markets. These movers and shakers of the markets could be big Investment Banks, Hedge Fund with a lot of cash, Financial Institutions, Fund Of Funds or Potential Investors with a lot of cash available to them.

Further we found the diffusion constant or volatility to be inversely proportional to the mass of the free particle. Free particles in space are somehow subject to Brownian motion and the diffusion or volatility formula found above indicates that macroscopic bodies do not exhibit such a behavior. Edward Nelson [17] in his work to derive the Schroedinger equation from Newton's second law of motion assumed the diffusion parameter to be inversely proportional to the mass multiplied by a constant.

if E is the average energy of the free particle that is time independent then within the context of the market r also is the time independent interest rate for a given trading transaction. As stated above, the American call option is issued to the stock holder at a fixed and time independent interest rate for a given strike price K that expires at time t.

In order to obtain the expected future payoff at the money that is the continuous measurable quantum state with corresponding and fixed interest rates we have to solve the time independent Shcroedinger equation (19).

The total expected future payoff given by equation (7) is defined in terms of the solution to the time independent Schroedinger equation $|V>$ that is a measurable quantum state

Since (for convenience we will drop the quantum notation for V):

$$Y\,(t,\omega) = V\,(X\,(t,\omega))\,e^{rt}$$

We will solve the time independent Schroedinger equation (19) in terms of the constant diffusion parameter that is the volatility of the stock movement. Further, we will obtain the relationship between interest rate and other market parameters.

We guess the solution to be given in the form of $V\,(x) = e^{\lambda x}$ where $\lambda > 0$ and is constant. By substituting $V\,(x) = e^{\lambda x}$ into (19) and solving for the parameter λ that is:

$\lambda^2 = -\frac{r}{D} \Rightarrow \lambda = \pm i\sqrt{\frac{r}{D}}$, where D is the diffusion

constant given in terms of the volatility that is: $D\,(\sigma) = \frac{1}{2}\sigma^2$ by substituting λ into $V\,(x) = e^{\lambda x}$ we will obtain $Y\,(t,\omega)$ in terms of the volatility of the market σ, then

$$Y\,(x,\omega) = A[sin\sqrt{\frac{r}{D}}x]e^{rt} \tag{20}$$

We have found the expected future payoff of the call option to be an oscillatory and stationary function with respect to the price movement for very low interest rates for a short period of time; this would make the exponential discount factor to be 1. Since the solution to the Schroedinger equation is a measurable quantum state with a given probability so is the total expected payoff of the call option given to the stock holder. Another word (20) is a measurable and square integrable quantum state given in terms of the price movement of the stock and A is called the normalization constant.

The aim of this work is to find the expected future payoff of the American call option at the money by solving the time independent Schroedinger equation. Now that the solution to the time independent Schroedinger equation is known and we also know from equation (9) that the supremum (*sup.*) of the expected future payoff at the money when $X\,(t=\tau)=K, \quad \tau$ is a stopping time or Markov time is defined by:

$$V\,(X) = SupE\|X\,(\tau)-K,0\,|\|X\,(\tau)=K\ =0$$

From equation (9) or the equation given above we have imposed a boundary condition to the expected future payoff at a certain stopping time $t=\tau$ to vanish. Since we have hedged the Bachelier type of the Black-Scholes equation given by (17) in order to obtain the time independent Schroedinger equation then the solution to the time independent Schroedinger should vanish at the money that would give the total expected future payoff of the American call option to vanish at$X\,(t=\tau)=K$: from equation (20) we will obtain the following:

$$Y\,(x,\omega) = A[sin\sqrt{\frac{r}{D}}x]e^{rt}\,|x\,(\tau)=K\ =0, \quad A\left(e^{rt}\to 1\right)\neq 0$$

$$\Leftrightarrow sin\sqrt{\frac{r}{D}}K=0 \Rightarrow \sqrt{\frac{r}{D}}K=n\pi$$

$\Rightarrow r=\left(\frac{D}{K^2}\right)n^2\pi^2$, where $n=1,2,3..$ is the number of times the stock holder buys a stock

$$\Rightarrow r_n=\left(\frac{\sigma^2}{2K^2}\right)n^2\pi^2 \tag{21}$$

For lower interest rates $e^{rt}\to 1$, the normalization constant can be found by normalizing (20) since it is a square integrable function with this probability density $P=|Y_n\,(x,\omega)|$. We have found interest rate r given in the time independent Shroedinger equation in terms of the volatility and the exercise price K for the American call option at the money (state b).

We have obtained quantized interest rates at the money when the stock price equals to the strike price K. At the second boundary when$x=0$, there is no expected payoff for the stock holder. Usually, options are issued to individuals who have bought stocks in a given company and the company wants to protect them against fluctuations of the stock price. Another way to understand this is that if there is no investment in a company there is no option (call or put) for you.

5. Summary

We have assumed in this work that the stock holder has entered into an American call option contract and this gives him the right to exercise when the stock price is above the strike price K. In order to model the stock price movement we used the Bachelier model. We assumed the stock price to move it way up from deep out of the money through state b where $X(t) = K$ where the expected payoff of the stock holder to be zero. We called $X(t) = K$ a point of singularity just because the expected payoff vanishes there. In order to find a quantum solution at the singularity, we used stopping time or Markov time and martingale theories to obtain a conditional expectation on the expected future payoff. Then we introduced Ito calculus to obtain a Bachelier-type of the Black-Scholes equation which we hedged by allowing $\Delta \to 0$. The hedging is called delta neutral and we compared the hedged Bachelier-type of the Black-Scholes equation to the time independent Schroedinger equation in Quantum Mechanics. Further we found the hedged Bachelier-type of the Black-Scholes equation to be identical to the time independent Schroedinger equation for a free particle in a box with infinite walls. Hence, from the comparison we were able to identify the volatility of the market as reciprocal of mass of the system meaning that we found the volatility to be approximately and inversely proportional to the mass (inertia of the stock price) [17]. One of the situations when many investors get very frustrated by in buying (selling) stocks is the waiting time, especially when one is dealing with a company with a very low volume of trading transactions. From the volatility formula (18), it is indicated that for very low transaction volume comes high volatility and vice-versa. We then solved the time independent Schroedinger equation to obtain the expected future payoff $Y(x, \omega)$ given by equation (20) that is an oscillatory and square integrable quantum state with a probability density of the expected future payoffs,$P = |Y_n(x, \omega)|$. We imposed the boundary condition $X(t) = K$ (singularity point or at the money) on equation (20) in order to solve for the interest rate. We finally found the interest in terms of the volatility of the market and the strike price K to be quantized and constant for each expected future payoff. Since interest rate is constant and fixed during the time the option is issued, the expected future payoffs are stationary and constant for low interest rates in very short period estimated in milliseconds. From our knowledge, this is the first time such an approach has been taking into account hedging the Bachelier's-type of the Black-Scholes equation at the money while the stock holder is awaiting for the stock price to move deep in the money. This approach is also valid for a stock holder with an American put option contract which means the stock holder is going to exercise only when the strike price is greater than the stock price $X(t) < K$ given by equation (4).

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