

Riemann Zeta Function and Hydrogen Spectrum

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Abstract: Significant analytic and numerical evidence, as well as conjectures and ideas connect the Riemann zeta function with energy-related concepts. The present paper is devoted to further extension of this subject. The problem is analyzed from the point of view of geometry and physics as wavelengths of hydrogen spectrum are found to be in one-to-one correspondence with complex-valued positions. A Zeta Rule for the definition of the hydrogen spectrum is derived from well-known models and experimental evidence concerning the hydrogen atom. The Rydberg formula and Bohr's semiclassical quantization rule are modified. The real and the complex versions of the zeta function are developed on that basis. The real zeta is associated with a set of quantum harmonic oscillators with the help of relational and inversive geometric concepts. The zeta complex version is described to represent continuous rotation and parallel transport of this set within the plane. In both cases we derive the same wavelengths of hydrogen spectral series subject to certain requirements for quantization. The fractal structure of a specific set associated with $\zeta(s)$ is revealed to be represented by a unique box-counting dimension.

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1. Rydberg Formula and Bohr Model

The emission spectrum of hydrogen can be expressed in terms of the Rydberg constant for hydrogen R_H using the Rydberg formula, namely

$$\frac{1}{\lambda} = R_H \left(\frac{1}{\tau_1^2} - \frac{1}{\tau_2^2} \right), \quad (1)$$

where τ_1 and τ_2 are integers such that $\tau_1 < \tau_2$. This spectrum is divided into a number of spectral series where for each one, $\tau_1 \geq 1$ and $\tau_2 = \tau_1 + 1, \tau_1 + 2, \tau_1 + 3, \dots$

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Initially the value of the constant is determined empirically.

In [1] and [2] Bohr shows that its value can be calculated from fundamental constants. In his planetary model an electron follows circular orbits around the nucleus of the hydrogen atom and these positions of the electron are associated with discrete energy levels. A nonclassical quantization rule arises from the implementation of Planck's theory of radiation [3] and Einstein's explanation of the photoelectric effect [4]. In terms of the angular momentum of the electron of mass m_e moving in an orbit of radius r_τ and with a speed v_τ with respect to the nucleus it can be expressed as

$$m_e v_\tau r_\tau = \tau \frac{h}{2\pi} = \tau \hbar, \quad \tau = 1, 2, 3, \dots \quad (2)$$

Bohr writes the energy associated with radiation as the difference between the energies of two levels $\Delta E = -(E_{final} - E_{initial}) = E_i - E_f$. Thus the Rydberg formula $\frac{1}{\lambda} = R_H \left(\frac{1}{\tau_1^2} - \frac{1}{\tau_2^2} \right)$ is defined with the same value and dimension of the Rydberg constant. The emission spectrum is derived from the Rydberg formula in the form $\frac{1}{\lambda_e} = R_H \left(\frac{1}{\tau_{final}^2} - \frac{1}{\tau_{initial}^2} \right)$ and is defined by the frequencies of radiation emitted by the atom when the electron from a state of being removed from the nucleus enters into reaction with it and is further moved to lower energy levels. The indexation τ_1 for the final state is fixed and the indexation τ_2 for the initial state varies from infinity to $\tau_1 + 1$. When the atom is exposed to certain forms of energy $\Delta E = (E_{final} - E_{initial}) = E_f - E_i$ the electron can be excited or moved to higher energy levels even to the state of a free particle. The positive sign in front of the brackets shows that this process is associated with the absorption of radiation. The indexation τ_1 for the initial state is fixed and the indexation τ_2 for the final state varies and the Rydberg formula takes the form $\frac{1}{\lambda_a} = R_H \left(\frac{1}{\tau_{initial}^2} - \frac{1}{\tau_{final}^2} \right)$. Thus for fixed τ_1 , τ_2 varies from $\tau_1 + 1$ to infinity.

Bound states of the hydrogen atom are defined by a discrete spectrum for $\tau_2 \ll \infty$. For values of $\tau_2 \rightarrow \infty$ electron's free state is defined which is the state of an isolated particle in case the electron is moved far apart from the nucleus by the absorption of photons. This state is characterized by a continuous spectrum.

The requirement for conservation of energy implies that the quantities related to the emission of radiation due to the descent of an electron from some initial state to a lower one are equal to quantities related to the absorption of radiation which causes the raise of the same electron from the same lower state to the same initial state.

2. Hydrogen Spectrum and Zeta Function

From the Rydberg formula for hydrogen one can derive the following relation

$$\zeta_H(s = 2, \tau_2) = \sum_{\tau=1}^{q+\tau_1} \frac{1}{\tau^2} = \sum_{\tau=1}^{\tau_1} \frac{1}{\tau^2} + \frac{q}{\tau_1^2} - \frac{1}{R_H} \sum_{i=1}^q \frac{1}{\lambda_i} \quad (3)$$

where q is the integer number of observed wavelengths in each of the series of spectral lines. The value of two of the variable s naturally arises from Bohr's energy-related

concept. From the wavelengths of hydrogen as compiled from [5], [6] and [7] in the following table (in nanometers):

Lyman	Balmer	Paschen	Brackett	Pfund	Humphreys
121.5	656.3	1875.1	4050	7460	12400
102.3	481.6	1281.8	2630	4650	7500
97.2	434.1	1093.8	2170	3740	5910
94.9	410.2	1004.9	1940	3300	5130
93.7	397.0	954.6	1820	3040	4670
93.0	388.9	922.9	1460	2280	3280
92.6	383.5	901.5			
92.3	364.6	886.3			
92.1		875.0			
91.9		866.5			
		866.5			

(4)

one can easily find that the terms of the consecutive $q + \tau_1$ sums of the zeta function differ insignificantly from the consecutive sums of the terms $\sum_{\tau=1}^{\tau_1} \frac{1}{\tau^2} + \frac{q}{\tau_1^2} - \frac{1}{R} \sum_{i=1}^q \frac{1}{\lambda_i}$ for all series of the hydrogen spectrum. Here is an example for the Balmer series:

$\zeta_H(2)$ -terms	λ_i (meters)	$\sum_{\tau=1}^2 \frac{1}{\tau^2} + \frac{q=1,\dots,8}{4} - \frac{1}{R_H} \sum_{i=1}^q \frac{1}{\lambda_i}$
1.00000		
1.25000		
1.36111	6.56E-07	1.36115
1.42361	4.86E-07	1.42369
1.46361	4.34E-07	1.46376
1.49139	4.10E-07	1.49161
1.51180	3.97E-07	1.51207
1.52742	3.89E-07	1.52776
1.53977	3.84E-07	1.54014
1.54977	3.65E-07	1.54020

(5)

This incomplete zeta function (since summation is performed up to a term) is a func-

tion of the sum of all observed wavelengths for a given series. Thus it can be associated with a quantum operator of Hamiltonian nature corresponding to the total energy of the bound state, the hydrogen atom in this case. The complete version of the zeta function for $\tau_2 \rightarrow \infty$ should be associated with an operator which includes the case with a free electron and thus the atom's unbound state.

One can deduce that the spatial frequency formulation of the spectrum (emission or absorption) of hydrogen follows a well-defined Zeta Rule, as far as a general nucleus-electron configuration is assumed.

3. Generalized Spectrum and Zeta Function

A general version of the zeta function can be developed as

$$\zeta(s, \tau_2) = \sum_{\tau=1}^{q+\tau_1} \frac{1}{\tau^s} = \sum_{\tau=1}^{\tau_1} \frac{1}{\tau^s} + \frac{q}{\tau_1^s} - \frac{1}{C} \sum_{i=1}^q \frac{1}{\lambda_i}, \quad (6)$$

where q is the number of wavelengths of some spectral series defined with respect to τ_1 and C is a constant. This constant equals the Rydberg constant in the case of hydrogen and Bohr's semiclassical electron-proton relation is encoded in it. For other configurations it can take different values. For example, for objects with Planck's mass and charge the constant calculated in a similar way appears to be of the order of the inverse Planck's length, namely

$$C_{Pl} = \frac{1}{4\pi} \frac{1}{l_{Pl}}. \quad (7)$$

To ensure the continuation of the parameter s to values other than $s = 2$ we write the Rydberg formula in a modified way as

$$\frac{1}{\lambda} = C \left[\frac{1}{(n_1^k)^2} - \frac{1}{(n_2^k)^2} \right], \quad (8)$$

where n_1 and n_2 are integers, k is a real number and thus $s = 2k$. Bohr's quantization rule becomes $mvr = n^k \hbar$ and one arrives at the original for $n^k = \tau$ or $k = \frac{\log \tau}{\log n}$. We can assume that for all other values of k infinitely many wavelengths can be derived and if they really exist, they should constitute some background generalized spectrum. For fixed values of the parameter k the zeta function can be written as

$$\zeta(s = 2k, n_2) = \sum_{n=1}^{q+n_1} \frac{1}{n^s} = \sum_{n=1}^{n_1} \frac{1}{n^s} + \frac{q}{n_1^s} - \frac{1}{C} \sum_{i=1}^q \frac{1}{\lambda_i}. \quad (9)$$

Generalization of the Rydberg formula is found in [8] where at the end Ritz writes "... the magnetic field in an atom may be regarded in all spectra as produced by two poles of opposite sign, which separately may occupy different positions in the atom. In hydrogen, these points lie at equal distances on a straight line. ...".

This idea prompts us to develop a

4. Circle - circle Intersection Model

We take four parameters d, n_1, n_2 and k and define the intersection of two circles where the circle C_l

$$x^2 + y^2 = \left(\frac{d}{n_2^k}\right)^2 \quad (10)$$

centered at the origin intersects the circle C_r

$$\left(x - \frac{2d}{n_1^k}\right)^2 + y^2 = \left(\frac{d}{n_2^k}\right)^2. \quad (11)$$

We set n_1, n_2 - integers with $n_1 < n_2$ and k - real. This is a pair of disjoint circles since $\frac{1}{2} \frac{2d}{n_1^k} > \frac{d}{n_2^k}$. The place of intersection along the x - coordinate is found to be $x = \frac{d}{n_1^k}$ through subtraction of both equations and solving for x . Upon substitution and solving for y one gets the relation

$$y^2 = \left(\frac{d}{n_2^k}\right)^2 - \left(\frac{d}{n_1^k}\right)^2. \quad (12)$$

Further we write

$$y^2 = -d^2 \left(\frac{1}{n_1^{2k}} - \frac{1}{n_2^{2k}}\right) = (\pm id)^2 \left(\frac{1}{n_1^{2k}} - \frac{1}{n_2^{2k}}\right) \quad (13)$$

and thus

$$y = \pm id \sqrt{\frac{1}{n_1^{2k}} - \frac{1}{n_2^{2k}}} \text{ or } \mp iy = d \sqrt{\frac{1}{n_1^{2k}} - \frac{1}{n_2^{2k}}}. \quad (14)$$

In order to preserve the dimensionless geometric model we set $-d^2 = C_R$, where C_R is a constant equal to the numerical value of the Rydberg constant. In addition, the presence of the negative sign is a good argument to associate this model with the negative energy content of bound states within the process of emission of radiation. For $k = 1$ one finds the value of the square of the vertical extension y (iy) equivalent to the wave numbers and thus to the wavelengths of interest for the hydrogen atom such that

$$\frac{1}{\lambda_e} = y^2 \Rightarrow \lambda_e = \frac{1}{y^2}. \quad (15)$$

In the same way all quantities previously expressed by the Rydberg formula and related to the spectral lines of the hydrogen series can be derived and plotted. The zeta function with respect to (12) takes the form

$$\zeta(s = 2k, n_2) = \sum_{n=1}^{q+n_1} \frac{1}{n^{2k}} = \sum_{n=1}^{n_1} \frac{1}{n^{2k}} + \frac{q}{n_1^{2k}} - \frac{1}{C_R} \sum_{i=1}^q y_i^2 = \sum_{n=1}^{n_1} \frac{1}{n^s} + \frac{q}{n_1^s} - \frac{1}{C_R} \sum_{i=1}^q \frac{1}{\lambda_i}. \quad (16)$$

We shall refer to the above pattern as the trivial case of intersection.

An important benefit from the circle-circle intersection concept is that complex numbers and complex-valued positions are introduced and they are related to real physical phenomena and observations through the correspondence

$$\frac{1}{\sqrt{\lambda}} \leftrightarrow \pm id \sqrt{\frac{1}{n_1^{2k}} - \frac{1}{n_2^{2k}}}. \quad (17)$$

Wavelengths of hydrogen spectrum can be derived from the imaginary part of the complex numbers

$$z_{1,2} = \frac{d}{n_1^k} \pm id \sqrt{\frac{1}{n_1^{2k}} - \frac{1}{n_2^{2k}}} \leftrightarrow \frac{C_R}{n_1^k} \pm iC_R \sqrt{\frac{1}{n_1^{2k}} - \frac{1}{n_2^{2k}}}, \quad (18)$$

for $k = 1$.

Therefore, the number of conjugate pairs of complex values $z_{1,2}$ equals the number of possible quantum energy states (levels) of the hydrogen atom.

We have defined so far the zeta summation operator as a function of real physical observables. These real values appear to be in one-to-one correspondence with certain complex quantities.

5. The Riemann Zeta Function and Spectral Interpretations

One can refer to the Riemann zeta function in light of the Hilbert-Polya conjecture. From Odlyzko's home page Polyá's statement reads "... *if the nontrivial zeros of the Xi-function were so connected with the physical problem that the Riemann hypothesis would be equivalent to the fact that all the eigenvalues of the physical problem are real.*". Another plausible version of this conjecture as summarized by Derbyshire in [9] is related to the possibility that the imaginary parts of the zeros $\frac{1}{2} + it$ of Riemann's $\zeta(s = \frac{1}{2} + it) = 0$ correspond to eigenvalues of some unbound Hermitian operator, i.e. these are the eigenvalues of some Hermitian matrix.

Montgomery's pair correlation conjecture [10] arises from the investigation on the correlation between pairs of zeros with respect to their imaginary parts. It closely follows from a function which is initially recognized by Dyson as Conrey tells in [11] as the pair correlation function for eigenvalues of large random Hermitian matrices. This function is discussed in relation with the distribution of energy levels in large systems of particles. The truth of the Riemann Hypothesis is assumed, i.e. only the imaginary parts of the non-trivial zeros are considered.

Another platform for discussion, in case the truth of the hypothesis is assumed is to write $\zeta[\frac{1}{2} \pm i(t \pm i\varepsilon)]$ for $|\frac{1}{2}| > \varepsilon$ and because of the conventional operation $\zeta[(\frac{1}{2} \pm \varepsilon) \pm it]$ to try to prove that either $\varepsilon = 0$, or $t \pm i\varepsilon$ is entirely real.

Following the ideas from the circle-circle intersection model with respect to the Riemann zeta function and the energy-related concepts cited above we have to distinguish between the complex valued positions $z_{1,2} = \frac{d}{n_1^k} \pm id \sqrt{\frac{1}{n_1^{2k}} - \frac{1}{n_2^{2k}}}$ and the complex numbers $s = \sigma \pm it$ as the parameter of the Riemann zeta function $\zeta(s)$. The former $z_{1,2}$ values are derived from the intersection of two disjoint circles and they are directly related to real energy values through their imaginary parts which have $\varepsilon = 0$ for all cases of the trivial intersection. The relation to some real physical problem of $s = \sigma \pm it$ and thus of the non-trivial zeros $s = \frac{1}{2} \pm it$ of the complex $\zeta(s)$ appears to be different in nature and shall be considered accordingly.

The geometry of the circle-circle intersection concept shows that the value of the real part of $z_{1,2}$ cannot be uniquely defined. It depends on the arbitrary choice of the position

of the intersecting couple of circles with respect to the origin. The most one can get is the relation $\frac{d}{n_1^k} - a = \varsigma$, where the separation between the centers of both circles remains fixed and equal to $\frac{2d}{n_1^k}$, the value a defines the arbitrary deflection from the origin of the circle C_l and ς is the unknown value of the real part of $z_{1,2}$. At the same time, the imaginary parts of $z_{1,2}$ do not depend on the position of the pair of intersecting circles with respect to the origin. They are a function of the separation between the centers of the circles and their radii. Thus the correspondence of the imaginary parts to energy levels remains valid for all values of the deflection a and thus ς . The values of the imaginary parts are irrelevant to any preferred choice of the origin.

In order to discuss the variable $s = 2k$ and the contribution of the new parameter k we fix the parameters d, n_1 and n_2 and we preserve the relation $n_1 < n_2$ as in the original Rydberg representation.

For $k > 0$ and thus $n_1^k < n_2^k$ the circles remain disjoint and the intersection of the circles is said to be in the complex plane as defined by expression (14). Wavelengths related to the emission spectrum are derived from this model. For $n_2 \rightarrow \infty$ the radii of circles C_l and C_r approach zero and both objects can be seen as point-circles and thus points. The complex valued positions $z_{1,2} = \frac{d}{n_1^k} \pm id\sqrt{\frac{1}{n_1^{2k}} - \frac{1}{n_2^{2k}}}$ become equal to $z_{1,2} = \frac{d}{n_1^k} \pm i\frac{d}{n_1^k}$ and they define the intersection between two points of zero size. This configuration represents the state of a free electron and is unique for the case with the emission of radiation.

The requirement for the conservation of energy is considered in a different way in the circle-circle intersection model. The positions and the roles of the divisors n_1 and n_2 are exchanged. Thus the circle C'_l

$$\left[x' - \left(x - \frac{d}{n_2^k}\right)\right]^2 + (y')^2 = \left(\frac{d}{n_1^k}\right)^2 \quad (19)$$

centered at $x - \frac{d}{n_2^k}$ intersects the circle C'_r

$$\left[x' - \left(x + \frac{d}{n_2^k}\right)\right]^2 + (y')^2 = \left(\frac{d}{n_1^k}\right)^2 \quad (20)$$

centered at $x + \frac{d}{n_2^k}$. The coordinate extensions are primed for convenience and they coincide with the x, y -axes. The separation between the two centers is equal to $\frac{2d}{n_2^k}$ and the radii are equal to $\frac{d}{n_1^k}$. The place of intersection along the x' -coordinate is found to be equal to $x = \varsigma$. Upon substitution and solving for y' one gets the relation

$$(y')^2 = \left(\frac{d}{n_1^k}\right)^2 - \left(\frac{d}{n_2^k}\right)^2 \quad (21)$$

and thus

$$y' = \pm d\sqrt{\frac{1}{n_1^{2k}} - \frac{1}{n_2^{2k}}}. \quad (22)$$

For $d^2 = C_R$ and $k = 1$ the value of the square of the vertical extension y' is equal to the square of the imaginary part of the complex numbers $z_{1,2}$ and thus equivalent to the same

wave numbers but in this case they are related to the absorption spectrum for hydrogen such that

$$\frac{1}{\lambda_a} = (y')^2 \Rightarrow \lambda_a = \frac{1}{(y')^2}. \quad (23)$$

The relation to the absorption spectrum is represented by real intersection of circles.

We face two possibilities to define certain positions by complex notations. The first option are the $z_{1,2} = \varsigma \pm id\sqrt{\frac{1}{n_1^{2k}} - \frac{1}{n_2^{2k}}} = \varsigma \pm it$ complex values arising from the intersection of disjoint circles. The other one are the real parametric notations $x' = x = \varsigma$ and $y' = \pm d\sqrt{\frac{1}{n_1^{2k}} - \frac{1}{n_2^{2k}}} = \pm t$ which in the conventional Argand complex plane can take the same value and position as $x' \pm iy' = \varsigma \pm it = z_{1,2}$. Possibly, these notations are interchangeable unless deeper specification is needed for the physical interpretation.

For values $k < 0$ and thus $n_1^k > n_2^k$ one can further investigate the Riemann zeta function and especially $\zeta(s = -2k)$ where k is an integer and thus $s = -2k$ is a trivial zero of the function. From the intersection relation

$$y^2 = -d^2\left(\frac{1}{n_1^{-2k}} - \frac{1}{n_2^{-2k}}\right) = d^2(n_2^{2k} - n_1^{2k}) = \frac{1}{\lambda_a} \quad (24)$$

which is a result from the real intersection of the circles $x^2 + y^2 = \left(\frac{d}{n_2^{-k}}\right)^2$ and $(x - \frac{2d}{n_1^{-k}})^2 + y^2 = \left(\frac{d}{n_2^{-k}}\right)^2$ and for $d^2 = C_R$ one can write

$$\zeta(-2k, n_2) = \sum_{n=1}^{n_1} \frac{1}{n^{-2k}} + \frac{q}{n_1^{-2k}} + \frac{1}{C_R} \sum_{i=1}^q \frac{1}{\lambda_i}. \quad (25)$$

Following the above consideration we associate absorption of radiation to this real intersection of circles. A concept that the zeros of the Riemann zeta function should be interpreted as an absorption spectrum is developed by Connes in [12] and [13], where the imaginary values of the non-trivial zeros are interpreted as missing spectral lines.

The correspondence of emission and absorption representations require the existence of an emission-related construction to the construction represented by (24). With the exchange of the divisors n_1 and n_2 and the same place of intersection equal to x the circle C'_i

$$\left[x'' - \left(x - \frac{d}{n_2^{-k}}\right)\right]^2 + (y'')^2 = \left(\frac{d}{n_1^{-k}}\right)^2 \quad (26)$$

centered at $x - \frac{d}{n_2^{-k}}$ intersects the circle C''_r

$$\left[x'' - \left(x + \frac{d}{n_2^{-k}}\right)\right]^2 + (y'')^2 = \left(\frac{d}{n_1^{-k}}\right)^2 \quad (27)$$

centered at $x + \frac{d}{n_2^{-k}}$ where the x'' coordinate coincides with the x coordinate. The separation between the two centers is equal to $\frac{2d}{n_2^{-k}}$ and radii are equal to $\frac{d}{n_1^{-k}}$. Upon solving for y'' one gets the relation

$$(y'')^2 = d^2\left(\frac{1}{n_1^{-2k}}\right) - \left(\frac{1}{n_2^{-2k}}\right) = -d^2(n_2^{2k} - n_1^{2k}) \quad (28)$$

and thus the complex valued positions

$$y'' = \pm id\sqrt{n_2^{2k} - n_1^{2k}}. \quad (29)$$

We associate these positions with the emission of radiation and they define the same points as the ones related to the absorption of radiation.

If one considers the values $y^2 = d^2(n_2^{2k} - n_1^{2k})$ in the form

$$y^2 = n_2^{2k} d^2 \left[1 - \left(\frac{n_1}{n_2}\right)^{2k}\right] \quad (30)$$

for $n_2^{2k} = Z^2$, where Z is a positive integer greater than one and $\frac{n_1}{n_2}$ is a unit fraction $\frac{1}{\tau}$ and $d^2 = C_R \rightarrow R_H$ the relation

$$y^2 = Z^2 R_H \left[1 - \left(\frac{1}{\tau^k}\right)^2\right] = \frac{1}{\lambda_{e,a}} \quad (31)$$

for $k = 1$ is the extension of the Rydberg formula for the spectrum of hydrogen-like elements. Many-electron states or many-particle states are traced with formula (31) and closer relations are suggested by Montgomery's pair-correlation formula.

From inversive geometry we know that two disjoint circles can be mapped into two concentric circles. We build up an inversion of circles C_l and C_r with the complex numbers $z_{1,2} = \varsigma \pm id\sqrt{\frac{1}{n_1^{2k}} - \frac{1}{n_2^{2k}}}$ as the cornerstone of the model. These complex valued positions lie on the radical axis of circles C_l and C_r . Point $z_1 = \varsigma + id\sqrt{\frac{1}{n_1^{2k}} - \frac{1}{n_2^{2k}}}$ is chosen as the center of circle C_o which is orthogonal to the pair of disjoint circles and thus z_1 has the same power with respect to both circles. Circle C_o intersects the real line (the line of centers) at the limiting points of each pencil of circles determined by circles C_l and C_r . An inversion circle C_{inv} is drawn centered at the left limiting point L_l such that the extension ς of the real part of $z_{1,2}$ and the right limiting point L_r appear as inversion points with ς the inversion pole. The new circles C_l^c and C_r^c are concentric and both are centered at the inversion pole ς . In this picture the complex valued positions $z_{1,2}$ and the limiting points L_l and L_r always remain within the annulus of the concentric circles.

As previously pointed in this paragraph the number of complex conjugate pairs $z_{1,2}$ equals the number of possible quantum energy states. This number equals the number of possible pairs of concentric circles C_l^c and C_r^c . These pairs of circles can be considered as geometric representation of Bohr's model of electrons traveling in stationary orbits around the nucleus. At the same time if one endows the true spatial frequency dimension to the parameter d in the circle-circle intersection model instead of being only numerically equal to the Rydberg constant and scale it by some constant of angular momentum dimension one can define a position-momentum phase space. Then and as it is well-known concentric circles represent the phase plane portrait of harmonic oscillators $p^2 + x^2$. The number of such resonators equals the number of energy states. Berry and Keating in [14] and [15] present evidence that in the series $\zeta(\frac{1}{2} + \nu E_n) = 0$, the E_n quantities are energy levels. They describe a Riemann operator as the quantum counterpart of classical Riemann dynamics based on connections between the Riemann zeta function and the classical

hamiltonian $H_{cl}(x, p) = xp$. After certain considerations, the number of quantum levels $N(E)$ with energy less than E is found to be

$$N(E) = \frac{E}{2\pi} [\log(\frac{E}{2\pi}) - 1] + \frac{7}{8} + \dots, \quad (32)$$

which equals Riemann's estimation for the number of roots of $\xi(t)$ less than a given quantity [16].

Presumably from a connection between the expression for the classical momentum $p_{cl} = \frac{E}{x}$ and its quantum version $p_q = k\hbar = \frac{2\pi}{\lambda}\hbar = \frac{1}{\lambda}h$ one can write

$$\int \frac{1}{\lambda} \sim \frac{1}{h} (E \int \frac{1}{x} - c), \quad (33)$$

where c is some constant. The right hand side is the analogue of the formula Berry and Keating use to derive relation (32). The left hand side is an integration over some continuous spectrum of the wave number $\frac{1}{\lambda}$. The related series $\sum_q \frac{1}{\lambda_q}$ can be derived after suitable normalization. The number of energy states associated to the discrete spectrum of wavelengths λ can be identified by the average spacing between the terms of the series.

In the geometry of the concentric pairs of circles one can easily recognize the landscape for the analytic continuation of the definition of the integral

$$\Gamma(s)\zeta(s) = \int_0^\infty \frac{x^{s-1}}{e^x - 1} dx \quad (34)$$

into the complex plane as performed by Riemann. To avoid the discontinuity of the integrand at the origin Riemann considers the integral

$$\int_\gamma \frac{(-v)^{s-1}}{e^v - 1} dv. \quad (35)$$

where the contour γ is defined from positive infinity and slightly above the real axis to some $\delta > 0$, then counterclockwise around a circle with radius δ with center at the origin and then from δ back to $+\infty$, just below the real axis. Thus from the relation

$$2 \sin(\pi s)\Gamma(s)\zeta(s) = i \int_\infty^\infty \frac{(-v)^{s-1}}{e^v - 1} dv \quad (36)$$

Riemann deduces that "... This equation now gives the value of the function $\zeta(s)$ for all complex numbers s

In fact this continuation implies complex numbers of the form $v_{1,2} = x \pm i(\epsilon \rightarrow 0)$ as the constituents of the contour γ due to the technique for consideration of the branch cut of the argument function which provides the insight to the continuation of the variable s into the complex numbers. The contour γ can be associated with the trajectory of an electron. The branch $+\infty \rightarrow v_1 = x + i(\epsilon \rightarrow 0) \rightarrow \delta$ can be associated with the movement of the electron from a free state towards the nucleus and thus with certain emission processes. The branch $\delta \rightarrow v_2 = x - i(\epsilon \rightarrow 0) \rightarrow +\infty$ can be related to the

trajectory of the electron from a state close to the nucleus outwards and associated with absorption processes.

Then Riemann considers the integral in a negative sense around the specified domain. The evaluation of the integral can be done with the assistance of a new contour γ_c consisting of two concentric circles connected via a contour wall and the Residue Theorem. One can derive

$$\int_{\gamma_c} \frac{(-w)^{s-1}}{e^w - 1} dw = 2\pi i \sum_q \text{Res} \left[\frac{(-w)^{s-1}}{e^w - 1}, w = \pm q2\pi i \right], \quad (37)$$

where q is an integer. The values $\pm q2\pi i$ appear in conjugate pairs and represent the discontinuities of the integrand of (36), i.e. the poles of the integrand. Alongside this contour deformation one gets a second set of complex numbers of the form $w_{1,2} = 0 \pm iy$ where the extension y corresponds to the values of the poles and the real part equals zero because the common center of the concentric circles which constitute the contour γ_c is at the origin. The contour γ_c can be associated with the discrete circular orbits of the electron around the nucleus.

The left part of equation (37) can also be written as

$$\int_{\gamma_c} \frac{(-w)^{s-1}}{e^w - 1} dw = \int_{\gamma_c} \frac{dw}{\left[\frac{(-w)^{s-1}}{e^w - 1} \right]^{-1}} \quad (38)$$

and thus the poles occur as zeros of some function $g(1-s) = (-w)^{1-s}(e^w - 1)$ with the same number of zeros as the number of poles.

If we set

$$\pm q2\pi i = \pm i d \sqrt{\frac{1}{n_1^{2k}} - \frac{1}{n_2^{2k}}} \leftrightarrow \frac{1}{\sqrt{\lambda}} \quad (39)$$

and thus provide a relation between the value of the poles and the imaginary parts of $z_{1,2}$, the latter can be used in the derivation of integral (37).

For further discussion on the function we emphasize on the two sets of complex numbers $v_{1,2} = x \pm i(\epsilon \rightarrow 0)$ and $w_{1,2} = 0 \pm iy$ which support the analytic continuation of $\zeta(s)$. To extend the understanding about these complex quantities we describe another circle-circle intersection model between two disjoint circles with their line of centers apart from the coordinate axes x and y . Thus the intersection of the circle C_1

$$(x - a)^2 + (y - b)^2 = \left(\frac{d}{n_2^k}\right)^2 \quad (40)$$

with the circle C_2

$$(x - c)^2 + (y - m)^2 = \left(\frac{d}{n_2^k}\right)^2 \quad (41)$$

is investigated. The separation between the centers is $\sqrt{|c - a|^2 + |m - b|^2} = \frac{d}{n_1^k}$ and the parameters d, n_1, n_2 and k have the same meaning and relation as in the trivial case of intersection and $k > 0$. The condition $\sqrt{|c - a|^2 + |m - b|^2} > \frac{2d}{n_2^k}$ guarantees

that the circles are disjoint. One can identify the trivial case of intersection by setting $a = b = m = 0$ and the separation $\sqrt{|c - a|^2 + |m - b|^2}$ equals $c = \frac{2d}{n_1^k}$. It is assumed that the pair of disjoint circles from the trivial case is parallel transported to a different place in the plane, such that all relations deduced for the trivial case hold for the newly positioned pair of circles, including the possibility to inverse the pair of disjoint circles into a pair of concentric circles. Then we first solve for x and we get

$$x = \frac{c^2 + m^2 - a^2 - b^2}{2(c - a)} - y \frac{m - b}{c - a}. \quad (42)$$

We set $A = \frac{c^2 + m^2 - a^2 - b^2}{2(c - a)}$ and $B = \frac{m - b}{c - a}$ and thus

$$x = A - yB. \quad (43)$$

Then we plug the value of x in (40) and solve for y . Thus

$$y = \frac{(A - a)B + b}{B^2 + 1} \pm i \frac{\sqrt{4(B^2 + 1)[(A - a)^2 + b - (\frac{d}{n_2^k})^2] - [2(A - a)B - b]^2}}{2(B^2 + 1)} \quad (44)$$

and the complex-valued representation follows from the understanding that disjoint circles intersect in complex valued positions. We simplify by setting $C = \frac{(A - a)B + b}{B^2 + 1}$ and $D = \frac{\sqrt{4(B^2 + 1)[(A - a)^2 + b - (\frac{d}{n_2^k})^2] - [2(A - a)B - b]^2}}{2(B^2 + 1)}$. Then we solve for x by plugging the value of y in (42). Finally we get the complex values

$$\begin{aligned} x_{1,2} &= A - BC \mp iBD \\ y_{1,2} &= C \pm iD \end{aligned} \quad (45)$$

which define the places of intersection of circles C_1 and C_2 . One can refer to the trivial case by taking $B = 0$ and $C = 0$ and thus $x_{1,2} = A \mp i0 = \frac{c}{2} \mp i0 = \frac{d}{n_1^k} \mp i0$ and $y_{1,2} = 0 \pm iD = 0 \pm id\sqrt{(\frac{1}{n_1^k})^2 - (\frac{1}{n_2^k})^2}$. These are not exactly the same values as the values defined in the trivial case, where $y_{1,2} = \pm id\sqrt{(\frac{1}{n_1^k})^2 - (\frac{1}{n_2^k})^2}$ appears in a conjugate pair but this is not the case with the x - place of intersection. In addition, the zero values are not recognized in the trivial case and we write directly $z_{1,2} = x \pm iy = \frac{d}{n_1^k} \pm id\sqrt{(\frac{1}{n_1^k})^2 - (\frac{1}{n_2^k})^2}$. Otherwise we have to write

$$z_{1,2} = \left(\frac{d}{n_1^k} \pm i0\right) \pm i\left[0 \pm id\sqrt{(\frac{1}{n_1^k})^2 - (\frac{1}{n_2^k})^2}\right] \quad (46)$$

which following the usual rules has to be revealed as

$$z_{1,2} = \frac{d}{n_1^k} \mp d\sqrt{(\frac{1}{n_1^k})^2 - (\frac{1}{n_2^k})^2} \pm i(0 + 0). \quad (47)$$

We lose the complex-valued position for the intersection between the disjoint circles which is the entity associated with the wavelengths and energy levels. A different value of the real part is derived and it defines the positions of the limiting points L_l and L_r .

To identify the true positions of the points of intersection along the radical axis of the disjoint circles and thus the position of the poles $z_{1,2}$ in the parallel transported complex we have to apply a rather inconvenient operation, as follows:

$$z_{1,2}^{int} = x_{1,2} \pm iy_{1,2} = [A - BC \mp iBD(i)] \pm i[C \pm iD(i)] = [(A - BC) \pm BD] \pm i(C \mp D). \quad (48)$$

The contradiction related to (46) is resolved in the same way by writing

$$z_{1,2} = \left[\frac{d}{n_1^k} \pm i0(i) \right] \pm i \left[0 \pm id \sqrt{\left(\frac{1}{n_1^k}\right)^2 - \left(\frac{1}{n_2^k}\right)^2} \right] = \frac{d}{n_1^k} \mp id \sqrt{\left(\frac{1}{n_1^k}\right)^2 - \left(\frac{1}{n_2^k}\right)^2}. \quad (49)$$

If one reveals the complex number $z_{1,2} = x_{1,2} \pm iy_{1,2}$ in the conventional way

$$z_{1,2}^{pos} = (A - BC \mp iBD) \pm i(C \pm iD) = [(A - BC) \mp D] \mp i(BD \mp C) \quad (50)$$

one arrives at the positions of the limiting points which still lie on the line of centers of the transported pair of disjoint circles.

A similar and unusual consideration appears in the treatment of the real intersection between circles in case their line of centers does not coincide with the x or y coordinates. The relation $\sqrt{|c - a|^2 + |m - b|^2} < \frac{2d}{n_2^k}$ guarantees that the circles get real intersection. The solution for the value of y is real and it is equal to

$$y = \frac{(A - a)B + b}{B^2 + 1} \pm \frac{\sqrt{[2(A - a)B - b]^2 - 4(B^2 + 1)[(A - a)^2 + b - (\frac{d}{n_2^k})^2]}}{2(B^2 + 1)} \quad (51)$$

and we set $D' = \frac{\sqrt{[2(A - a)B - b]^2 - 4(B^2 + 1)[(A - a)^2 + b - (\frac{d}{n_2^k})^2]}}{2(B^2 + 1)}$. Thus we get the real parametric values

$$\begin{aligned} x_{1,2} &= A - BC \mp BD' \\ y_{1,2} &= C \pm D' \end{aligned} \quad (52)$$

for the intersection of the pair of circles. We can define the intersecting points as $z_1^{int} = (x_1, y_1) = (A - BC - BD', C + D')$ and $z_2^{int} = (x_2, y_2) = (A - BC + BD', C - D')$, which are the same as the complex values for the poles $z_{1,2}^{int}$ in (48). We cannot get further information about positions along the line of centers as it is possible with equation (50) for the case of disjoint circles. To do this we have to represent the positions of the real intersection from (52) in complex notation as in (45) and then apply the operations from (48) and (50).

We shall refer to both possibilities of this model as the general case of intersection.

From the above considerations we come to the conclusion that the complex values $v_{1,2} = x \pm i0$ and $w_{1,2} = 0 \pm iy$ are special cases of some $v'_{1,2} = x \pm i\alpha$ and $w'_{1,2} = y \pm i\beta$

which define the extended coordinate axes of a complex plane S . The general case of intersection between circles and points along the radical axis of these objects is defined within this complex plane by the complex numbers

$$s_{1,2}^{int} = v'_{1,2} \pm iw'_{1,2} = [x \pm i\alpha(i)] \pm i[y \pm i\beta(i)] = (x \mp \alpha) \pm i(y \mp \beta). \quad (53)$$

The position of the limiting points in the case of circle-circle intersection or the points themselves in the case of intersection between point-circles along the line of centers is defined by the complex values

$$s_{1,2}^{pos} = v'_{1,2} \pm iw'_{1,2} = (x \pm i\alpha) \pm i(y \pm i\beta) = (x \mp \beta) \pm i(y \pm \alpha). \quad (54)$$

The general case ranges over all positions between two circular objects with respect to each other and the position of the pair itself with respect to the conventional real and imaginary axes. The real and the imaginary parts of the extensions $x \pm i\alpha$ and $y \pm i\beta$ of these axes need not be orthogonal as in the traditional Cartesian representation of complex numbers. For example, the intersection of the disjoint circles $(x - a)^2 + (y - b)^2 = r^2$ and $(x - a)^2 + (y + b)^2 = r^2$ such that $b > r$ is at $x = a \pm i\sqrt{b^2 - r^2}$ and $y = 0$. The circles are identical and they are placed at equal distance from the real axis, above and below it. The intersection is along the radical axis between them and it coincides with the real line. Thus the orientation of the imaginary parts $\pm i\sqrt{b^2 - r^2}$ is parallel to the real line, i.e. parallel to the segment which defines the extension of the associated real part.

The assumption that the pair of circles involved in the trivial case of intersection is parallel transported to a new position can be accomplished through suitable affine transformation composed of certain rotations, translations and scaling. The complex number

$$z_{inv}^{pole} = \frac{d}{n_1^k} (\cos \phi + i \sin \phi) = \zeta (\cos \phi + i \sin \phi) \quad (55)$$

represents a ϕ -radians counterclockwise rotation of the place of intersection $x = \frac{d}{n_1^k} = \zeta$ or the rotation of the inversion pole ζ as the center of the associated pair of concentric circles. Circle C_l remains centered at the origin and the new position of the center of circle C_r is

$$z_{C_r}^{center} = \frac{2d}{n_1^k} (\cos \phi + i \sin \phi) = 2\zeta (\cos \phi + i \sin \phi) \quad (56)$$

with both (55) and (56) lying on the line of centers of both circles. The position of the intersection between circles C_l and C_r remains along the radical axis of the circles which is identified with the y -coordinate in the trivial case of intersection. It is always perpendicular to the line of centers of the circles which is identified with the x -coordinate in the trivial case. Our purpose is to define relations, the same or similar to those in the trivial case of intersection which correspond to wave numbers and thus to wavelengths after the affine transformation. In addition, we have to take care about the physical interpretation of the circle-circle intersection, i.e. whether it refers to the emission spectrum or to the absorption spectrum with the former being represented by complex-valued positions and

the latter being represented by the same but real-valued positions. We multiply the real vectors $(x, y_{1,2}) = (\varsigma, \pm d\sqrt{\frac{1}{n_1^{2k}} - \frac{1}{n_2^{2k}}})$ with the counterclockwise rotation matrix

$$\mathbf{R}(\phi) = \begin{pmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{pmatrix} \quad (57)$$

to get the proper positions of the real intersection. Since they are associated with absorption we leave them in the real parametric form

$$\begin{aligned} x' &= \varsigma \cos \phi \mp d\sqrt{\frac{1}{n_1^{2k}} - \frac{1}{n_2^{2k}}} \sin \phi \\ y' &= \varsigma \sin \phi \pm d\sqrt{\frac{1}{n_1^{2k}} - \frac{1}{n_2^{2k}}} \cos \phi. \end{aligned} \quad (58)$$

One can write (58) in the usual complex notation by

$$z_{1,2}^{int} = (\varsigma \cos \phi \mp d\sqrt{\frac{1}{n_1^{2k}} - \frac{1}{n_2^{2k}}} \sin \phi) + \imath(\varsigma \sin \phi \pm d\sqrt{\frac{1}{n_1^{2k}} - \frac{1}{n_2^{2k}}} \cos \phi) \quad (59)$$

but this does not provide the most comprehensive description of the complex-valued intersection as interpretation of emission related quantities. As previously mentioned one cannot identify the positions of the limiting points along the line of centers. For values of $n_2 \rightarrow \infty$ one gets intersection between point-circles which is only associated with the emission spectrum. Last but not least, the affine transformation of the complex-valued positions appears to be sensitive to the application of the usual matrix rotation. We multiply the complex-valued vector $z_1^{int} = \varsigma + \imath d\sqrt{\frac{1}{n_1^{2k}} - \frac{1}{n_2^{2k}}}$ by the counterclockwise rotation matrix and we get

$$\begin{aligned} x' &= \varsigma \cos \phi - \imath d\sqrt{\frac{1}{n_1^{2k}} - \frac{1}{n_2^{2k}}} \sin \phi \\ y' &= \varsigma \sin \phi + \imath d\sqrt{\frac{1}{n_1^{2k}} - \frac{1}{n_2^{2k}}} \cos \phi. \end{aligned} \quad (60)$$

Then we write for the new position

$$z_1 = (\varsigma \cos \phi - \imath d\sqrt{\frac{1}{n_1^{2k}} - \frac{1}{n_2^{2k}}} \sin \phi) + \imath(\varsigma \sin \phi + \imath d\sqrt{\frac{1}{n_1^{2k}} - \frac{1}{n_2^{2k}}} \cos \phi). \quad (61)$$

When we reveal expression (61) in the usual way we get

$$z_1^{pos} = (\varsigma \cos \phi - d\sqrt{\frac{1}{n_1^{2k}} - \frac{1}{n_2^{2k}}} \cos \phi) + \imath(\varsigma \sin \phi - d\sqrt{\frac{1}{n_1^{2k}} - \frac{1}{n_2^{2k}}} \sin \phi) \quad (62)$$

and thus the position of the limiting point L_l which lies on the line of centers is defined instead of the true position of intersection which lies on the radical axis. For $n_2 \rightarrow \infty$ one gets the position of the complex zero $z_1^{pos} = 0 + \imath 0$. Since we have applied counterclockwise

rotation in (56) then the position of the limiting point L_r and the center of circle C_r represented by a point-circle for $n_2 \rightarrow \infty$ has to be in the same direction. Contrary to this we have to apply multiplication of the complex vector z_2^{int} by the clockwise rotation matrix and write the complex-valued position in the form

$$z_2^{pos} = (\varsigma \cos \phi + id \sqrt{\frac{1}{n_1^{2k}} - \frac{1}{n_2^{2k}}} \sin \phi) - i(-\varsigma \sin \phi + id \sqrt{\frac{1}{n_1^{2k}} - \frac{1}{n_2^{2k}}} \cos \phi) \quad (63)$$

to get the proper position

$$z_2^{pos} = (\varsigma \cos \phi + d \sqrt{\frac{1}{n_1^{2k}} - \frac{1}{n_2^{2k}}} \cos \phi) + i(\varsigma \sin \phi + d \sqrt{\frac{1}{n_1^{2k}} - \frac{1}{n_2^{2k}}} \sin \phi) \quad (64)$$

which for $n_2 \rightarrow \infty$ becomes $z_2^{pos} = 2\varsigma(\cos \phi + i \sin \phi)$, i.e. it is the same as (56). To evaluate the proper positions of the intersection in complex form we multiply the complex vector $z_1^{int} = \varsigma + id \sqrt{\frac{1}{n_1^{2k}} - \frac{1}{n_2^{2k}}}$ by the matrix

$$\mathbf{R}'(\phi) = \begin{pmatrix} \cos \phi & \sin \phi \\ \sin \phi & -\cos \phi \end{pmatrix} \quad (65)$$

and we apply the operation from (53) such that

$$\begin{aligned} z_1^{int} &= [\varsigma \cos \phi + id \sqrt{\frac{1}{n_1^{2k}} - \frac{1}{n_2^{2k}}} (\sin \phi)(i)] + i[\varsigma \sin \phi - id \sqrt{\frac{1}{n_1^{2k}} - \frac{1}{n_2^{2k}}} (\cos \phi)(i)] = \\ &[\varsigma \cos \phi - d \sqrt{\frac{1}{n_1^{2k}} - \frac{1}{n_2^{2k}}} \sin \phi] + i[\varsigma \sin \phi + d \sqrt{\frac{1}{n_1^{2k}} - \frac{1}{n_2^{2k}}} \cos \phi]. \end{aligned} \quad (66)$$

Then we multiply the complex vector $z_2^{int} = \varsigma - id \sqrt{\frac{1}{n_1^{2k}} - \frac{1}{n_2^{2k}}}$ by the same matrix $\mathbf{R}'(\phi)$ and with the operation from (53) we write

$$\begin{aligned} z_2^{int} &= [\varsigma \cos \phi + (-id \sqrt{\frac{1}{n_1^{2k}} - \frac{1}{n_2^{2k}}}) (\sin \phi)(i)] + i[\varsigma \sin \phi + (-id \sqrt{\frac{1}{n_1^{2k}} - \frac{1}{n_2^{2k}}}) (-\cos \phi)(i)] = \\ &[\varsigma \cos \phi + d \sqrt{\frac{1}{n_1^{2k}} - \frac{1}{n_2^{2k}}} \sin \phi] + i[\varsigma \sin \phi - d \sqrt{\frac{1}{n_1^{2k}} - \frac{1}{n_2^{2k}}} \cos \phi]. \end{aligned} \quad (67)$$

The complex-valued positions (66) and (67) are generalized by (53) and the real value needed to be associated with the wave numbers of interest is defined as

$$\alpha^2 + \beta^2 = \frac{1}{\lambda} \Rightarrow \lambda = \frac{1}{\alpha^2 + \beta^2}, \quad (68)$$

where

$$\alpha^2 + \beta^2 = (\mp d \sqrt{\frac{1}{n_1^{2k}} - \frac{1}{n_2^{2k}}} \sin \phi)^2 + (\pm d \sqrt{\frac{1}{n_1^{2k}} - \frac{1}{n_2^{2k}}} \cos \phi)^2 = d^2 \left(\frac{1}{n_1^{2k}} - \frac{1}{n_2^{2k}} \right). \quad (69)$$

The position of the inversion pole ς where the radical axis crosses the line of centers and where the associated pair of concentric circles is centered can be written as

$$z_{inv}^{pole} = x \pm iy \quad (70)$$

in the notations from (53) and within the ϕ -radians rotation as

$$z_{inv}^{pole} = \varsigma \cos \phi \pm \varsigma \sin \phi. \quad (71)$$

6. The Complex Version

With the assistance of the above described models and considerations we discuss the Riemann zeta function complex version $\zeta(s)$ for $s = \sigma \pm it$.

In Bohr's sense this is a modification of the concept of the angular momentum into a complex-valued representation where $k = \sigma \pm it$ and $mvr = n^k \hbar = n^{\sigma \pm it} \hbar$. This is a subject largely discussed in Regge theory and more details can be found in [17] and in [18]. Regge poles which can be of interest from the point of view of the present survey are defined as singularities in the complex angular momentum plane. In general Riemann type of continuation for positive integer values of the angular momentum to the complex plane is applied in relation to wave amplitudes and particles' spectrum.

By analogy with the circle-circle intersecting patterns the complex-valued zeta function in the form $\zeta[(\sigma \pm it)^2]$ can be derived from the expression for some y -intersection module where the term $k = \sigma \pm it$ is directly plugged in such that

$$\zeta[(\sigma \pm it)^2, n_2] = \sum_{n=1}^{n_1} \frac{1}{(n^{\sigma \pm it})^2} + \frac{q}{(n_1^{\sigma \pm it})^2} \pm \frac{1}{d^2} \sum_{i=1}^q y_i^2. \quad (72)$$

The square of the parameter $s = \sigma \pm it$ appears according to the relation with the energy concepts as derived in Bohr's sense for the real version of zeta and one can define the $\zeta(\sigma \pm it)$ version for $\frac{k}{2} = \frac{\sigma \pm it}{2}$ and thus $mvr = n^{\frac{k}{2}} \hbar = n^{\frac{\sigma \pm it}{2}} \hbar$. The relation to the hydrogen spectrum is supported by the assumption $\mp d^2 = C_R$.

Complex numbers are not an ordered set and we cannot define a relation between the quantities with the indexes n_1 and n_2 as divisors. We apply directly the analogy from the real version. With $k = \sigma - it$ we write

$$y_e^2 = -d^2 \left[\frac{1}{(n_1^{\sigma - it})^2} - \frac{1}{(n_2^{\sigma - it})^2} \right] \quad (73)$$

for the expected complex intersection between disjoint circles which is associated with the emission spectrum. The term $\frac{1}{(n_1^{\sigma - it})^2}$ is related to linear separation and the term $\frac{1}{(n_2^{\sigma - it})^2}$ is related to some radial extension. For the real intersection and its relation to the absorption spectrum we write

$$y_a^2 = d^2 \left[\frac{1}{(n_1^{\sigma - it})^2} - \frac{1}{(n_2^{\sigma - it})^2} \right] \quad (74)$$

where the term $\frac{1}{(n_1^{\sigma - it})^2}$ is related to some radial extension and the term $\frac{1}{(n_2^{\sigma - it})^2}$ is related to linear separation.

In both cases the expected place of intersection along the x -coordinate axis is

$$x = \frac{d}{n_1^{\sigma - it}} = \frac{d}{n_1^\sigma} e^{it \log n_1} = \frac{d}{n_1^\sigma} (\cos t \log n_1 + i \sin t \log n_1) \quad (75)$$

which represents the extended version of the x -axis as described in (53). For $\phi = t \log n_1$ this is exactly the new place of intersection along the line of centers of two circular objects as expressed by the rotation in (55). Note that for $n_1 = 1$ or $t = 0$ the intersection is

$$x = \frac{d}{n_1^\sigma} + i0 \quad (76)$$

and for $n_1 \rightarrow 1$ and $t \rightarrow 0$ it becomes $x = \frac{d}{n_1^\sigma} + i(\epsilon \rightarrow 0)$. These are the same type of complex values as $v_1 = x + i0$ and $v_1 = x + i(\epsilon \rightarrow 0)$ which arise from the contour deformation in (35).

Equation (73) in its full version reads

$$y_e = \pm i d \sqrt{\left(\frac{\cos 2t \log n_1}{n_1^{2\sigma}} - \frac{\cos 2t \log n_2}{n_2^{2\sigma}}\right)^2 + \left(\frac{\sin 2t \log n_1}{n_1^{2\sigma}} - \frac{\sin 2t \log n_2}{n_2^{2\sigma}}\right)^2} \\ \left(\cos \frac{1}{2} \arctan \frac{\frac{\sin 2t \log n_1}{n_1^{2\sigma}} - \frac{\sin 2t \log n_2}{n_2^{2\sigma}}}{\frac{\cos 2t \log n_1}{n_1^{2\sigma}} - \frac{\cos 2t \log n_2}{n_2^{2\sigma}}} + i \sin \frac{1}{2} \arctan \frac{\frac{\sin 2t \log n_1}{n_1^{2\sigma}} - \frac{\sin 2t \log n_2}{n_2^{2\sigma}}}{\frac{\cos 2t \log n_1}{n_1^{2\sigma}} - \frac{\cos 2t \log n_2}{n_2^{2\sigma}}}\right) \quad (77)$$

and we simplify it by writing

$$y_e = \pm i d \sqrt{M} (\cos \varphi + i \sin \varphi) = \mp d \sqrt{M} (\sin \varphi - i \cos \varphi). \quad (78)$$

Respectively, we simplify the extended version of (74) as

$$y_a = \pm d \sqrt{M} (\cos \varphi + i \sin \varphi). \quad (79)$$

The modulus M can be developed into

$$M = \sqrt{\left(\frac{1}{n_1^{2\sigma}}\right)^2 + \left(\frac{1}{n_2^{2\sigma}}\right)^2 - \frac{1}{n_1^{2\sigma}} \frac{1}{n_2^{2\sigma}} 2 \cos(2t \log n_1 - 2t \log n_2)} \quad (80)$$

which as a form of the law of cosines defines certain distances. For $(2t \log n_1 - 2t \log n_2) = 2k\pi$ with $k = 0, 1, 2, 3, \dots$ it equals

$$M = \sqrt{\left(\frac{1}{n_1^{2\sigma}} - \frac{1}{n_2^{2\sigma}}\right)^2}. \quad (81)$$

Then, upon substitution in (78) we obtain

$$y_e = \mp d \sqrt{\frac{1}{n_1^{2\sigma}} - \frac{1}{n_2^{2\sigma}}} (\sin \varphi - i \cos \varphi) \quad (82)$$

which precisely equals the value from the trivial case of intersection after rotation as defined in (66) and (67) for $\sigma = 0$. These are the same type of complex values as $w_{1,2} = 0 \pm iy$ which we associate with the poles in the integral evaluation in (37) since for $\varphi = 2k\pi$ and $k = 0, 1, 2, 3, \dots$ we can write

$$y_e = 0 \pm i d \sqrt{\frac{1}{n_1^{2\sigma}} - \frac{1}{n_2^{2\sigma}}}. \quad (83)$$

From (76) and (83) and since in the trivial case of intersection zeros are not directly revealed we can write

$$z_{1,2}^{int} = \frac{d}{n_1^\sigma} \pm id \sqrt{\frac{1}{n_1^{2\sigma}} - \frac{1}{n_2^{2\sigma}}} \quad (84)$$

for the positions of intersection in the absence of rotation. These complex-valued positions are the familiar ones from the trivial case and thus

$$\mathfrak{S}_{z_{1,2}^{int}}^2 = d^2 \left(\frac{1}{n_1^{2\sigma}} - \frac{1}{n_2^{2\sigma}} \right) = \frac{1}{\lambda_e} \Rightarrow \lambda_e = \frac{1}{\mathfrak{S}_{z_{1,2}^{int}}^2}. \quad (85)$$

When rotation is considered we have the necessary formulation for the rotation of the x -place of intersection (75) which is recognized as the representation of the extended x -axis. We have to consider the representation of the y -axis in the form (83), i.e. a rotation around the origin instead of a certain displacement which can define an affiliation of this vector to the parallel transported pair of objects through the affine transformations (66) and (67) with the requirement for uniform rotation

$$\frac{1}{2} \arctan \frac{\frac{\sin 2t \log n_1}{n_1^{2\sigma}} - \frac{\sin 2t \log n_2}{n_2^{2\sigma}}}{\frac{\cos 2t \log n_1}{n_1^{2\sigma}} - \frac{\cos 2t \log n_2}{n_2^{2\sigma}}} = \varphi = t \log n_1. \quad (86)$$

The rotation of the inversion pole ζ is at φ -radians by definition and this is the angle between the complex vector z_{inv}^{pole} and the real line. The angle between (82) and the real line is $\varphi' = \arctan \frac{\cos \varphi}{\sin \varphi} = \arctan(\cot \varphi)$. Then $\varphi + \arctan(\cot \varphi) = \frac{\pi}{2}$ because of the trigonometric identity $\tan(\frac{\pi}{2} - \varphi) = \cot \varphi$ and thus $\pi - [\varphi + \arctan(\cot \varphi)] = \frac{\pi}{2}$. The complex vectors (82) are perpendicular to the complex vector z_{inv}^{pole} . Then they are parallel to the rotated and translated imaginary parts of the complex vectors (66) and (67) and equal in size. The wave numbers and wavelengths of interest derived in (68) and (69) for the parallel transported and rotated complex are the same as these calculated in (85), namely

$$\frac{1}{\lambda_e} = d^2 \left(\frac{1}{n_1^{2k}} - \frac{1}{n_2^{2k}} \right). \quad (87)$$

Thus for $\varphi = t \log n_1$ parallelism and identities between points, vectors and trajectories are preserved as a function of some continuous parameter t .

As previously stated the expression for the real intersection does not supply satisfactory information about the behavior of the parallel transported complex. Equation (79) is similar to the expression for the positions of the limiting points or the centers of the circular objects as derived in (63) and (64) for $\sigma = 0$ instead of being of the same type as the complex-valued intersection. However, we can benefit from this inconvenience since generally

$$\frac{d}{d\varphi} y_a = y_e \quad (88)$$

and thus the derivative of a representation related to position equals the representation related to wave numbers and correspondingly to momentum.

One can introduce discrete values of t related to the familiar values of wave numbers as

$$t_n = d \sqrt{\frac{1}{n_1^{2k}} - \frac{1}{n_2^{2k}}} \quad (89)$$

where certain unique amount of rotation and unique positions of objects from (75) and (82) are obtained. The same can be defined for the affine transformation (66) and (67) for $\phi = \varphi = t_n \log n_1$. In addition, relation to eigenvalues can be revealed for $\log n_1 = \phi'$ and we can, for example in (79) write

$$y_a = d \sqrt{\frac{1}{n_1^{2\sigma}} - \frac{1}{n_2^{2\sigma}}} e^{i\phi' d \sqrt{\frac{1}{n_1^{2\sigma}} - \frac{1}{n_2^{2\sigma}}}} = t_n e^{it_n \phi'}. \quad (90)$$

For $\varphi \neq t \log n_1$ the requirement for uniform rotation is not fulfilled but parallelism and equality still hold between vectors which can be associated with the (66), (67) and (82) vectors. This is equivalent to a rotation of the real vectors $(0, t_n)$ and (ς, t_n) performed with the matrix

$$\mathbf{R}''(\varphi, t, n_1) = \begin{pmatrix} \cos t \log n_1 & -\sin \varphi \\ \sin t \log n_1 & \cos \varphi \end{pmatrix}. \quad (91)$$

Within this rotation the vector $(0, t_n)\mathbf{R}''$ is not perpendicular to the vector $(\varsigma, 0)\mathbf{R}''$ but it is still parallel and equal to the $(\varsigma, t_n)\mathbf{R}''$ vector. If such an affine transformation is considered from the perspective of the trivial case of intersection, i.e. the positions are considered before the rotation it represents the intersection of two circles placed above and below the real line but centered at different positions from the origin with the x -place of intersection still equal to ς . This configuration does not affect the value of the positions of intersection along the radical axis between the objects and thus the relation to wavelengths.

Similar consideration about discreteness and continuity can be applied for the values of M from (80). Besides the familiar discrete values (81) for $(2t \log n_1 - 2t \log n_2) = 2k\pi$ another set of unique discrete values of this metric can be obtained for the possible values of t_n from (89). For $(t \log n_1 - t \log n_2) \neq k\pi$ and $t \neq \text{constant}$ the complex-valued positions (78) and (79) represent wave-like trajectories and the same can be assigned to the affine transformed positions (66) and (67) following the considerations for equality and parallelism.

Generally, representation (73) is developed as a combination of contours which define the analytic continuation of zeta. The zeta function for s -complex variable is itself derived from (73) where the relations to wave numbers and wavelengths and thus to energy states are preserved. $\zeta(s)$ remains a function of the sum of observed wavelengths within a given series of the hydrogen spectrum for $\mp d^2 = C_R \rightarrow R_H$ and for specific discrete values of the other parameters involved. Otherwise, it can serve to represent infinitely many values and positions within some general continuum.

7. Fractals

The generalized formula for the volume of a sphere can be written as

$$V = \frac{2}{s} \frac{\pi^{\frac{s}{2}}}{\Gamma(\frac{s}{2})} R^s \quad (92)$$

and the formula for the related surface can be written as

$$A = 2 \frac{\pi^{\frac{s}{2}}}{\Gamma(\frac{s}{2})} R^{s-1}, \quad (93)$$

where R is the size of the radius of some sphere of dimension s .

One can re-write the functional relation

$$\pi^{-\frac{s}{2}} \Gamma(\frac{s}{2}) \zeta(s) = \pi^{-\frac{1-s}{2}} \Gamma(\frac{1-s}{2}) \zeta(1-s) \quad (94)$$

which is satisfied by the Riemann zeta function in the form

$$\frac{\Gamma(\frac{s}{2})}{\pi^{\frac{s}{2}}} \sum_R \frac{1}{R^s} = \frac{\Gamma(\frac{1-s}{2})}{\pi^{\frac{1-s}{2}}} \sum_R \frac{1}{R^{1-s}} = \frac{\Gamma(\frac{1-s}{2})}{\pi^{\frac{1-s}{2}}} \sum_R R^{s-1} \quad (95)$$

for $n = R_n$. Without loss of generality we can re-arrange (95) into

$$2 \frac{\pi^{\frac{1-s}{2}}}{\Gamma(\frac{1-s}{2})} \sum_{R_n} \frac{1}{R_n^s} = 2 \frac{\pi^{\frac{s}{2}}}{\Gamma(\frac{s}{2})} \sum_{R_n} R_n^{s-1}. \quad (96)$$

The right hand side is exactly a summation over surfaces of spheres defined by integer-sized radii R_n . The variable s is clearly endowed with a meaning of dimension with $(s-1)$ - the dimension of the surfaces. The summation over volumes of spheres of dimension s and with radii $(\frac{1}{R_n})$ can be derived from

$$2 \frac{\pi^{\frac{1-s}{2}}}{\Gamma(\frac{1-s}{2})} \sum_{R_n} \frac{1}{R_n^s} = H(s) \frac{2}{s} \frac{\pi^{\frac{s}{2}}}{\Gamma(\frac{s}{2})} \sum_{R_n} \left(\frac{1}{R_n}\right)^s, \quad (97)$$

where $H(s) = s \pi^{\frac{1-2s}{2}} \frac{\Gamma(\frac{s}{2})}{\Gamma(\frac{1-s}{2})}$.

Then we recall that Riemann by making use of the equation $\int_0^\infty e^{-nx} x^{s-1} dx = \frac{\Gamma(s)}{n^s}$ derives the integral $\Gamma(s) \zeta(s) = \int_0^\infty \frac{x^{s-1}}{e^x - 1} dx$. Instead of a linear transformation of the x coordinate into nx with n - integer we consider this modification as a contour deformation into the length of an arc $n\theta \rightarrow R\theta$ of radius R and θ is the subtended angle by the arc. The new variable of the integrand of

$$\int_0^\infty \frac{(nx)^{s-1}}{e^{nx}} d(nx) = \int_0^\infty \frac{(R\theta)^{s-1}}{e^{R\theta}} d(R\theta) \quad (98)$$

is thus the length of an arc which extends into a circle for $\theta \geq 2\pi$ and this consideration is again about contours of the γ_c -type (37) related to the analytic continuation of the

zeta function. A set of consecutive concentric circles is defined with $R_n = 1, 2, 3, \dots$. In relation to the functional equation and the considerations (96) and (97) we can extend these contours with another set of circles with radii equal to the unit fractions $\frac{1}{R_n}$ for $R_n = 1, 2, 3, \dots$. If the former set of concentric circles extends outwards from the unit circle for $R = n = 1$, the latter extends inwards from the unit circle and towards the origin and includes the discontinuity at the origin. Then we shall consider these two sets of concentric circles as fractals, since concentricity satisfies basic requirements for the definition of fractals, namely the self-similarity property. Fractals are used to model a number of physical problems like differences between densities, potential differences and attractors, i.e. objects and events which are related to the energy-concept discussed in this paper. The concept of interest to the present survey is the box-counting dimension of the fractal set $E = \{0\} \cup \{\frac{1}{n} : n = 1, 2, 3, \dots\}$ which is

$$D_{box}(E) = \frac{1}{2}. \quad (99)$$

This set can be associated with the above described set of concentric circles with radii equal to the unit fractions and extending inwards from the unit circle towards and including the zero at the origin. Since the dimension s of some fractal set E can be defined as

$$s = \lim_{size \rightarrow 0} \frac{\log bulk(E)}{\log size(E)} \quad (100)$$

which is a representation of Theiler in [19], then in relation to (99) we can write

$$(size(E))^{\frac{1}{2}} = \left(\frac{1}{R_n}\right)^{\frac{1}{2}} = (bulk(E)). \quad (101)$$

This appears to be a special case mostly because of the inclusion of the terms $\frac{1}{R_n}\theta$ in the interpretation of the functional equation and the contours associated with the zeta function. The inclusion of the zero at the origin makes the $s = \frac{1}{2}$ box-counting dimension unique. One can treat (97) in a different manner and to re-arrange it as

$$2 \frac{\pi^{\frac{1-s}{2}}}{\Gamma(\frac{1-s}{2})} \sum_{R_n} \frac{1}{R_n^s} = H'(s) \frac{1}{2\pi^{\frac{s}{2}}} \sum_{R_n} \frac{1}{R_n^s}, \quad (102)$$

where $H'(s) = \frac{4\pi^{\frac{1}{2}}}{s\Gamma(\frac{1-s}{2})\Gamma(\frac{s}{2})}$. This is a summation over the inverse volumes of spheres with radii R_n which can be physically interpreted as summation over certain amounts of pressure.

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