What is a Quantum Equation of Motion?

Ali Shojaei-Fard*

Institute of Mathematics,University of Potsdam,
Am Neuen Palais 10, D-14469 Potsdam, Germany

Received 09 October 2012, Accepted 31 December 2012, Published 15 January 2013

Abstract: We apply combinatorial Dyson-Schwinger equations (in the context of the renormalization Hopf algebra) and noncommutative differential calculus to present a new interpretation from quantum motions.

Keywords: Connes-Kreimer Hopf Algebra of Feynman Graphs; Dyson-Schwinger Equations; Theory of Connections
PACS (2010): 03.70.+k; 11.10.-z; 03.65.Fd; 03.65.-w; 02.10.-v

Introduction

When Connes and Kreimer could discover a connection between renormalizable Quantum Field Theory and the Riemann-Hilbert problem in the context of the renormalization Hopf algebra of Feynman diagrams, some new advanced approaches to non-perturbative Quantum Field Theory have been addressed which can be classified in two general separate frameworks.

According to the first framework which is nicely formulated by Connes and Marcolli [5, 6], there is a method for the consideration of non-perturbative theory underlying the Riemann-Hilbert problem where we should apply the techniques of summation of divergent series modulo functions with exponential decrease of a certain order. This method which is called Borel summability provides a new description from non-perturbative phenomena under the local wild fundamental group. The Borel summation treatment in quantum field theory is well-known as a method for evaluating divergent formal series $\widehat{f}(g)$ in the coupling constants. In some theories, the related formal Borel transformation $S\widehat{B}\widehat{f}(g)$ is convergent with the property that the function $S\widehat{B}\widehat{f}(g)$ has singularities on the positive real axis which reflect the non-perturbative effects. By applying this Borel

* Email: shojaei-fard@math.uni-potsdam.de, shojaeifa@yahoo.com

summation method to the formal series $g^+_{eff}(0)$ (the effective coupling) of the renormalized perturbative theory, Connes-Marcolli could initiate a Tannakian formalism in the study of Quantum Field Theory.

The second framework which is algorithmically formulated by Kreimer and his colleagues ([1, 2, 14, 15, 16, 17, 18, 21, 22, 23]) considers a new version of Dyson-Schwinger equations in terms of the Connes-Kreimer renormalization Hopf algebra and Hochschild cohomology theory. This approach provides new combinatorial techniques to solve these equations which report about some non-perturbative information.

Thanks to these two viewpoints and also, Dubois-Violette results on noncommutative differential calculus [10, 11], in this letter we want to show that why Dyson-Schwinger equations address quantum motions.

1. Dyson-Schwinger Equations and Quantum Motions

Perturbative Quantum Field Theory considers some methods to eliminate ultraviolet and infrared (sub-)divergences of iterated ill-defined Feynman integrals in terms of Feynman diagrams. A Feynman diagram is a finite labeled oriented graph which contains some vertices for presenting interactions and some edges. The edges can be divided into two classes external edges which address elementary particles with assigned momenta and internal edges which present virtual particles. Renormalization allows us to remove these nested divergences from diagrams such that at the end of the day, the Lagrangian of the theory will be altered by adding counterterms related with removed (sub-)divergences.

1.1 Renormalization Hopf algebra

Perturbative renormalization in terms of Bogoliubov recursion and Zimmermann forest formula leads us to formulate a coproduct structure on Feynman diagrams of a renormalizable Quantum Field theory. This coproduct which is found by Kreimer [13] is given by

$$\Delta_{FG}(\Gamma) = \mathbb{I} \otimes \Gamma + \Gamma \otimes \mathbb{I} + \sum_{\gamma \subset_{\neq} \Gamma} \gamma \otimes \frac{\Gamma}{\gamma}, \quad \forall \Gamma \neq \mathbb{I} \tag{1}$$

such that the sum is over all disjoint unions of superficially 1PI Feynman proper subgraphs. Applying number of internal edges or number of independent loops is enough to determine an antipode map recursively which reports the presence of a Hopf algebraic structure H_{FG} on Feynman diagrams. It is a connected graded commutative non-cocommutative Hopf algebra such that thanks to the Milnor-Moore theorem, there is an alternative way to drive this Hopf algebra on the basis of the pre-Lie operator insertion on Feynman diagrams. [3, 7, 13]

The infinite dimensional complex Lie group $G(\mathbb{C})$ associated with H_{FG} (which is known as the Lie group of diffeographisms) is the main object for describing physical information on the basis of the Riemann-Hilbert problem where BPHZ renormalization (as a recursive step by step program for removing sub-divergences) supports the existence of a unique Birkhoff factorization for elements of $G(\mathbb{C})$. [3, 4]

The BPHZ formalism can be encoded by the commutative algebra A_{dr} of Laurent series with finite pole part equipped with a Rota-Baxter map R_{ms} given by

$$R_{ms}(\sum_{i\geq -m}^{\infty} c_i z^i) = \sum_{i\geq -m}^{-1} c_i z^i. \tag{2}$$

Thanks to the Atkinson theorem [8, 9], the Rota-Baxter algebra (A_{dr}, R_{ms}) (which addresses the multiplicativity of renormalization) provides a unique Birkhoff factorization for elements of the space $G(\mathbb{C}) = Hom_{\mathbb{C}}(H_{FG}, A_{dr})$ of characters. Now let us consider the space $Loop(G(\mathbb{C}), \mu)$ of loops γ_μ (depending on the mass parameter μ) on the infinitesimal punctured disk $\mathbf{\Delta}^*$ around $z = 0$ and with values in $G(\mathbb{C})$. The components of the Birkhoff factorization on elements of this loop space enable us to calculate counterterms, renormalized values, renormalization group and $\beta-$function. These components are computed based on deforming of the dimensionally regularized Feynman rules character underlying the antipode map and the map R_{ms}. Hence we can hope to interpret each unrenormalized theory in terms of a loop $\gamma_\mu \in Loop(G(\mathbb{C}), \mu)$ such that for each $z \in \mathbf{\Delta}^*$, $\gamma_\mu(z)$ is called dimensionally regularized Feynman rule character. After applying each Feynman rules character on a given amplitude, its related Green function will be determined. [3, 4, 5, 6]

Applying a decorated version of non-planar rooted trees leads us to obtain a universal simplified toy model from the renormalization Hopf algebra which is called Connes-Kreimer Hopf algebra and denoted by H_{CK}. Decorations allow us to modify this Hopf algebra with respect to a physical theory. Sub-divergences are vertices such that connected vertices show nested loops and vertices without any connection reports independent loops. This model guarantees the recursive nature of the renormalization coproduct (1) on the basis of the grafting operator B^+ which is helpful to study the pair (H_{CK}, B^+) as a universal object with respect to Hochschild cohomology theory. [7, 8, 9]

The grading structure defines naturally a canonical increasing filtration on $H = H_{FG}$ which produces a decreasing filtration on the dual space $L(H, A_{dr})$ (consisting of all linear maps from H to A_{dr}). It indicates a metric structure on this dual space such that together with the convolution product with respect to the coproduct (1), we obtain a complete filtered Rota-Baxter algebra $(L(H, A_{dr}), *, R_{ms})$ of weight one . [8, 9]

Recently, there exists some practical investigations about the application of this Hopf algebraic treatment in the study Quantum ElectroDynamics (QED)and Quantum ChromoDynamics (QCD). It is shown that important identities among Feynman graphs such as Ward identities in QED and Slavnov-Taylor identities in QCD can be encoded by some Hopf ideals of the Hopf algebra of Feynman diagrams $H_{FG}(QED)$ and $H_{FG}(QCD)$. This result is on the basis of rewriting the renormalization coproduct on 1PI Green's functions which leads us to explain perturbative treatments of quantum gauge theories in the language of the renormalization Hopf algebra. [15, 21, 22, 23]

1.2 Dyson-Schwinger Equations (DSEs) and Connections

From theoretical physicists point of view, DSEs apply to encode non-perturbative theory. The recursive nature of this class of equations allows us to reformulate them in the context of the renormalization coproduct which provides a process for passing from perturbation theory to non-perturbative phenomena. Indeed, if we consider the Connes-Kreimer Hopf algebra together with the grafting operator B^+ under the Hochschild cohomology theory, then a new approach to DSEs in the context of the renormalization coproduct investigates.

Let us consider the chain complex $(C = \bigoplus_{n\geq 0} C^n, \mathbf{b})$ on H with respect to the Connes-Kreimer coproduct such that C^n is the space of $n-$cochains which consists of all linear maps $L : H \longrightarrow H^n$ and the coboundary operator $\mathbf{b}$ is given by

$$\mathbf{b}L := (id \otimes L)\Delta_{FG} + \sum_{i=1}^{n} (-1)^i \Delta_i L + (-1)^{n+1} L \otimes \mathbb{I} \tag{3}$$

where Δ_i is the coproduct Δ_{FG} applied to the i-th factor in $H^{\otimes n}$. The cohomology of this complex which is denoted by $\mathrm{HH}^\bullet(H)$ is trivial for each $n \geq 2$. There is also a surjective map from $\mathrm{HH}^1(H)$ to the space $\mathrm{Prim}(H)$ which means that for each primitive 1PI Feynman graph Γ, the grafting operator $B^{+,\Gamma}$ determines a Hochschild one cocycle.

For a given family $\{B^{+,\gamma_n}\}_{n\geq 1}$ of Hochschild one cocycles, a combinatorial DSE is defined by

$$X = \mathbb{I} + \sum_{n=1}^{\infty} w_n B^{+,\gamma_n}(X^{n+1}) \tag{4}$$

such that w_n are scalars and γ_n are primitive Feynman graphs of loop order n. Applying the measure $\widetilde{\mu}$ which is identified by

$$\phi(B^{+,\gamma_n}(\mathbb{I})) = \int d\widetilde{\mu}_{\gamma_n} \tag{5}$$

(such that $\phi \in G(\mathbb{C})$ is the dimensionally regularized Feynman rules character) allows us to formulate the corresponding analytic version of these equations. Each equation DSE has a unique solution $(c_n)_{n\geq 0}$ determined by

$$c_n = \sum_{m=1}^{n} w_m B^{+,\gamma_m}\Big(\sum_{k_1+...+k_{m+1}=n-m, k_i \geq 0} c_{k_1} ... c_{k_{m+1}} \Big) \tag{6}$$

such that these elements play the role of the generators of a Hopf subalgebra H_{DSE} of H. [1, 2, 14, 16]

Here we are going to consider two processes which lead to indicate a family of connections with respect to each Dyson-Schwinger equation.

In one process which is based on Connes-Marcolli categorical framework and it is explained in [20], we can identify a family of flat equi-singular connection with respect to each equation DSE. These connections can be collected in a neutral Tannakian category which is recovered by the category $\mathrm{Rep}_{G^*_{\mathrm{DSE}}}$ of finite dimensional representations of the affine group scheme G^*_{DSE} related with Hopf subalgebra H_{DSE}. This category can be seen as a sub-category of the category Rep_{U^*} of finite dimensional representations of the affine group scheme U^* which has a universal property with respect to the Riemann-Hilbert correspondence. This procedure enables us to study Dyson-Schwinger equations at the level of the Connes-Marcolli universal Hopf algebra of renormalization H_U. The shuffle nature of H_U and its independence of physical theories address a more practical techniques for computing DSEs. This means that for a given equation DSE in a renormalizable theory Φ, we can lift it to H_U, solve the corresponding equation and then pull back the solution to the physical theory.[20]

Factorization of Feynman diagrams into primitive components at the level of solutions of DSEs leads us to translating information from perturbative quantum field theory to non-perturbative theory. Consider the equation

$$X = 1 + \sum_{f_n} w^n B^+_{f_n}(X^n) \tag{7}$$

in the shuffle type Hopf algebra H_U such that $\{f_n\}_{n\in\mathbb{N}}$ is the set of generators of H_U at the algebra level. Its unique solution can be formulated with respect to the $*-$Euler type factorization such that $*$ is the shuffle product of H_U. We have

$$X = \prod_{f_n}^{\star} \frac{1}{1 - w^n(f_n)} \tag{8}$$

such that (f_n) is a word with length one.

One important note is that the connections derived from the above process are related with the regularization method. In the second process which is the main purpose of this part, we will introduce a new family of connections associated with DSEs and independent of dimensional regularization.

For a given equation DSE, with the associated Hopf subalgebra H_{DSE} (depending on the unique solution of DSE), consider the unital associative noncommutative algebra $A_{\mathrm{DSE}} := (L(H_{\mathrm{DSE}}, \mathbb{C}), *)$ over $\mathbb{C}$ with the center $Z(A_{\mathrm{DSE}})$ and letting $\mathrm{Der}(A_{\mathrm{DSE}})$ be the $Z(A_{\mathrm{DSE}})-$module of all infinitesimal characters (i.e. derivations) over A_{DSE}.

Proposition 1..1. There is a differential calculus on A_{DSE} which depends upon the space of derivations. We call it the differential graded algebra (DGA) with respect to the equation DSE.

Proof 1..2. Let $\Omega^n_{\mathrm{Der}}(A_{\mathrm{DSE}})$ be the space of all $Z(A_{\mathrm{DSE}})-$multilinear anti-symmetric mappings from $\mathrm{Der}(A_{\mathrm{DSE}})^n$ into A_{DSE} by convention $\Omega^0_{\mathrm{Der}}(A_{\mathrm{DSE}}) = A_{\mathrm{DSE}}$. Set

$$\Omega^{\bullet}_{\mathrm{Der}}(A_{\mathrm{DSE}}) := \bigoplus_{n\geq 0} \Omega^n_{\mathrm{Der}}(A_{\mathrm{DSE}}). \tag{9}$$

For each $\omega \in \Omega^n_{\mathrm{Der}}(A_{\mathrm{DSE}})$ and $\theta_i \in \mathrm{Der}(A_{\mathrm{DSE}})$, its anti-derivation differential operator d_{DSE} of degree one (i.e. $d^2_{\mathrm{DSE}} = 0$) is given by

$$d_{\mathrm{DSE}}\omega(\theta_0, ..., \theta_n) :=$$

$$\sum_{k=0}^{n}(-1)^k\theta_k\omega(\theta_0,...,\widehat{\theta_k},...,\theta_n)+\sum_{0\leq r<s\leq n}(-1)^{r+s}\omega([\theta_r,\theta_s],\theta_0,...,\widehat{\theta_r},...,\widehat{\theta_s},...,\theta_n). \quad (10)$$

The information $(\Omega^{\bullet}_{\mathrm{Der}}(A_{\mathrm{DSE}}),d_{\mathrm{DSE}})$ determines the differential graded algebra related to DSE.

A $Z(A_{\mathrm{DSE}})-$bilinear anti-symmetric map ω in $\Omega^2_{\mathrm{Der}}(A_{\mathrm{DSE}})$ is called *non-degenerate*, if for any element $\phi\in A_{\mathrm{DSE}}$, there exists a unique derivation $\theta_\phi = ham(\phi)$ of A_{DSE} (i.e. *Hamiltonian vector field*) such that for each arbitrary derivation θ, $\omega(\theta_\phi,\theta)=\theta(\phi)$.

Remark 1..3. The differential graded algebra $\Omega^{\bullet}_{\mathrm{Der}}(A_{\mathrm{DSE}})$ can be applied to consider non-commutative symplectic structures which is useful to study integrable systems at the level of combinatorial Dyson-Schwinger equations.

Now using theory of connections on modules is enough to provide projective modules from Hamiltonian derivations.

Proposition 1..4. For a given equation DSE and measure $\widetilde{\mu}$ (identified by the equation 5),

(i) One can determine a connection ∇ with respect to DSE.

(ii) There is a natural Hermitian structure compatible with the connection ∇. For given arbitrary generators θ_ϕ,θ_ψ, it is given by

$$<\theta_\phi,\theta_\psi>:=\int\overline{\phi}\psi d\widetilde{\mu}.$$

(iii) The composition

$$\nabla^2:\mathrm{Der}_{\mathrm{Ham}}(A_{\mathrm{DSE}})\otimes_{Z(A_{\mathrm{DSE}})}\Omega^p(Z(A_{\mathrm{DSE}}))\longrightarrow\mathrm{Der}_{\mathrm{Ham}}(A_{\mathrm{DSE}})\otimes_{Z(A_{\mathrm{DSE}})}\Omega^{p+2}(Z(A_{\mathrm{DSE}}))$$

is $\Omega(A_{\mathrm{DSE}})-$linear and therefore its restriction to $\mathrm{Der}_{\mathrm{Ham}}(A_{\mathrm{DSE}})$ determines the curvature

$$k:\mathrm{Der}_{\mathrm{Ham}}(A_{\mathrm{DSE}})\longrightarrow\mathrm{Der}_{\mathrm{Ham}}(A_{\mathrm{DSE}})\otimes_{Z(A_{\mathrm{DSE}})}\Omega^2(Z(A_{\mathrm{DSE}}))$$

of the connection ∇.

Proof 1..5. (i) One can define a bundle

$$\mathrm{Der}_{\mathrm{Ham}}(A_{\mathrm{DSE}})\otimes_{\mathbb{C}}Z(A_{\mathrm{DSE}})\longrightarrow\mathrm{Der}_{\mathrm{Ham}}(A_{\mathrm{DSE}}) \quad (11)$$

on the free $Z(A_{\mathrm{DSE}})-$module $\mathrm{Der}_{\mathrm{Ham}}(A_{\mathrm{DSE}})$ generated by all Hamiltonian derivations of the initial algebra. There is a one to one correspondence between linear sections $s:\mathrm{Der}_{\mathrm{Ham}}(A_{\mathrm{DSE}})\longrightarrow\mathrm{Der}_{\mathrm{Ham}}(A_{\mathrm{DSE}})\otimes_{\mathbb{C}}Z(A_{\mathrm{DSE}})$ of this bundle and linear maps $\nabla:\mathrm{Der}_{\mathrm{Ham}}(A_{\mathrm{DSE}})\longrightarrow\mathrm{Der}_{\mathrm{Ham}}(A_{\mathrm{DSE}})\otimes_{Z(A_{\mathrm{DSE}})}\Omega^1(Z(A_{\mathrm{DSE}}))$ identified by

$$s=s_0+j\circ\nabla,\quad s(\eta)=\eta\otimes 1+j(\nabla\eta) \quad (12)$$

such that

$$j:\mathrm{Der}_{\mathrm{Ham}}(A_{\mathrm{DSE}})\otimes_{Z(A_{\mathrm{DSE}})}\Omega^1(Z(A_{\mathrm{DSE}}))\longrightarrow\mathrm{Der}_{\mathrm{Ham}}(A_{\mathrm{DSE}})\otimes_{\mathbb{C}}Z(A_{\mathrm{DSE}}) \quad (13)$$

is the usual injective map and

$$s_0(\eta) := \eta \otimes 1, \quad \eta \in \mathrm{Der}_{\mathrm{Ham}}(A_{\mathrm{DSE}}). \tag{14}$$

Now with attention to [10, 11, 19], the projective property of $\mathrm{Der}_{\mathrm{Ham}}(A_{\mathrm{DSE}})$ supports that ∇ is a connection.

(ii) Based on the measure $\widetilde{\mu}$, it is enough to consider the corresponding Hilbert space $L^2(L(H_{\mathrm{DSE}}, \mathbb{C}), \widetilde{\mu})$ such that here $L(H_{\mathrm{DSE}}, \mathbb{C})$ can be seen as a locally compact Hausdorff topological space with the topology induced by filtration.

Now consider the Hilbert space $\mathcal{H} := L^2(L(H_{\mathrm{DSE}}, \mathbb{C}), \widetilde{\mu})$ (depending on the equation DSE) and suppose $\mathcal{L}^1(\mathcal{H})$ be the trace class operators on $\mathcal{H}$. For each $T \in \mathcal{L}^1(\mathcal{H})$, define $\mathrm{Tr}(T) := \sum_{n \geq 0} < T\epsilon_n, \epsilon_n >$ such that $\{\epsilon_n\}_{n\geq 0}$ is an orthonormal basis for $\mathcal{H}$. Set

$$\Delta := -\mathrm{Tr} \circ \nabla \circ d_{\mathrm{DSE}} \tag{15}$$

such that ∇ is introduced with the proposition 1..4.

Corollary 1..6. Δ is the Laplacian operator related to the connection ∇ and the differential graded operator d_{DSE}.

There is a universal treatment for this construction. If we apply the universal differential forms ([10, 11, 19]), then they produce the universal differential calculus $\Omega_u(A_{\mathrm{DSE}})$ on A_{DSE} where the projective $Z(A_{\mathrm{DSE}})-$module $\mathrm{Der}_{\mathrm{Ham}}(A_{\mathrm{DSE}})$ determines the universal $\Omega_u(A_{\mathrm{DSE}})$-connection and its associated Laplacian operator.

Conclusion

So we could associate a new family of connections based on the unique solution of a DSE and independence of the regularization process. This class of connections provides this reasonable fact that DSEs report quantum equations of motion in a noncommutative differential geometric configuration which can be useful to understand the geometry of non-perturbative theory.

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